Name …………………………………… Course …………………………….
Print and use this sheet in conjunction with ***MathinSite***'s 'Complex Numbers' applet.

# Complex Numbers

## Theory Sheet 1
## Number Systems and their Graphical Representation

### Learning Outcomes
After using the ***MathinSite*** Complex Numbers applet and its accompanying tutorial and theory sheets (available from http://mathinsite.bmth.ac.uk/html/applets.html), you should

- be aware of how complex numbers arise
- be aware of the composition of a complex number in rectangular form
- be able to manipulate complex numbers in rectangular form using the basic operations +, -, × and ÷
- be aware of the composition of a complex number in polar form
- be able to convert complex numbers from rectangular to polar form and *vice versa*
- understand that complex numbers and 2-D vectors are essentially the same
- be able to manipulate complex number addition and subtraction graphically as vector addition and subtraction
- be able to manipulate complex numbers in polar form using the basic operations +, -, × and ÷
- be able to determine the $n^{th}$ power of a complex number
- be able to determine the $n^{th}$ roots of a complex number

### Prerequisites
Before using the applet, this theory sheet and any other accompanying sheets, familiarity with the following mathematics would be useful (but not vital).

- The solution of quadratic equations
- Two-dimensional vectors
- The rules of Indices (Powers)

However, ***even without this knowledge***, just using the applet can help in your appreciation of the mathematics involved.

### Background
This theory sheet, together with the accompanying tutorial sheets and applet, investigate complex numbers as studied in first-year undergraduate engineering (so, consequently, will use *j*, not *i*). Not only are complex numbers widely used in electrical engineering, where they can be used to solve basic a.c. electrical systems containing inductance, resistance and capacitance, but also, for example, in solving problems relating to forces in two-dimensional space (as a substitute for vectors).

The applet provides a graphical representation of some basic operations with complex numbers such as rectangular and polar form and conversion, addition and subtraction of complex numbers and the positioning of the roots of a complex number.

The applet/tutorial/theory sheet combination should help deepen your understanding of the meaning and manipulation of complex numbers – both graphically and mathematically.

**Number Systems** (a bit of a long section, but bear with me!)

Number Systems are a human invention. Almost as soon as you begin to talk, someone has you counting (toys, fingers and toes, boiled eggs, etc). The numbers used are called the ***Natural Numbers*** (or **counting numbers**) and are 1, 2, 3, 4, 5, etc (with zero sometimes included). Mathematicians denote this number system $\mathbb{N}$. Natural numbers can be visualised by counting objects (3 pens, 2 thumbs, 4 friends, etc.) or by spacing the numbers along in a line (with nothing in between the numbers):

. . . . . . .
0 1 2 3 4 5 …

The natural numbers are great for addition (one pen plus three pens gives four pens), and sometimes for subtraction (three pens take away one pen gives two pens). Unfortunately $\mathbb{N}$ is useless when it comes to a problem like 'three pens take away four pens'. How is this problem solved? By the mathematicians inventing a new number system - in this case the ***Integers***, denoted $\mathbb{Z}$. This number system contains all of the natural numbers but is extended to include (zero and) the negative numbers. Note that in real-world terms the integers still don't solve 'three pens take away four pens gives …', since what does 'minus one pen' look like? Even so, we are all happy to use and understand negative numbers. After all, we all have bank accounts! Note here then that 'minus one pen' is not a suitable visualisation of negative numbers; a suitable visualisation would be that of directed numbers (e.g. three steps forward and four steps back – in the opposite direction to forwards – leaves you one step back; or £200 in the bank followed by a £500 withdrawal leaves you 'owing' the bank £300). Following the same visualisation as above, the integers can be represented by:

. . . . . . .
… -2 -1 0 1 2 … (again with nothing in between)

Now, $\mathbb{Z}$ can cope with any problem involving addition and subtraction, but consider this. You and a friend have a bag containing 10 sweets. To share these out *equally* (one for you – one for me) would result in five sweets each. No problem. But supposing the bag contained 11 sweets? 'One for you – one for me' would result in five each and one left over, which in the name of equality would have to be split down the middle. OK, defined in terms of a number in $\mathbb{Z}$, how many do you each get? Perhaps you might say 'five and a bit'. But the 'bit' has no numerical representation. Now *we* all know about a 'half' – but $\mathbb{Z}$ doesn't. So we have to extend the number system again to include … let's call them ***fractions*** and write a 'half' as ½. This new number system, that contains all the fractions (one number written over another number), is called the ***Rationals***, denoted $\mathbb{Q}$, which necessarily contains all the numbers in $\mathbb{Z}$, which in turn contains all the numbers in $\mathbb{N}$.

——— ——— ——— ———
… -2 -1 0 1 2 …

(this now has values in between the integers e.g. ½, ¾, etc., but not enough to produce a continuous line! There are still gaps.)

So the rationals can solve all problems that involve fractions. Unfortunately there are some numbers that cannot be expressed as fractions (hence the gaps on the line above). You may have seen in your younger days $\pi$ written as 22/7. This was only ever an approximation, never exact. You can obtain better rational approximations for $\pi$, 355/113, for example, but no matter how good the fractional approximation, it will never equal $\pi$. The same can be said for other numbers; $\sqrt{2}$, also cannot be expressed as a fraction. So what do we do? Extend the number system again!

This time the ***decimals*** are incorporated and the resulting number system is called the ***Real Numbers***, denoted $\mathbb{R}$, which necessarily contains all the numbers in $\mathbb{Q}$, which in turn contains all the numbers in $\mathbb{Z}$, which in turn contains all the numbers in $\mathbb{N}$. So $\pi$ (and $\sqrt{2}$, etc.) can now be expressed as an, albeit infinitely long, decimal and the number line above now has the gaps filled to produce a continuous line.

…   -2   -1   0   1   2   …

and this is called the ***Real Number Line***.

Real number lines you use every time you draw a set of coordinate axes.

By the time you left school, this was probably as far as you went in terms of number systems. If you cast your mind back, though, can you remember a particular type of problem that 'did not have a solution'? In fact, you were probably even told, "This problem has no solution". This is half-right. Let's revisit some problems of that type and notice there comes a point when, *even armed with the reals*, you have to stop.

Solve $x^2 + 1 = 0$.
Here, simple rearrangement gives $x = \pm\sqrt{-1}$. Two solutions, $x = \sqrt{-1}$ and $x = -\sqrt{-1}$

Solve $x^2 + 9 = 0$.
Again, simple rearrangement gives $x = \pm\sqrt{-9} = \pm\sqrt{9}\times\sqrt{-1} = \pm 3\sqrt{-1}$

Solve $x^2 + 4x + 13 = 0$.
This is slightly more complicated, requiring the formula for solving a quadratic, i.e.

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} = \frac{-4 \pm \sqrt{16-52}}{2} = \frac{-4 \pm \sqrt{-36}}{2} = \frac{-4 \pm 6\sqrt{-1}}{2} = -2 \pm 3\sqrt{-1}$$

The stopping point, and hence the problem, is encountered each time when the square root of –1 appears.

**Note that the square root of any negative number, $\sqrt{-a}$, can always be written as $\sqrt{a}\times\sqrt{-1}$**

So, at school, hopefully you were told not just that these problems have "no solution", but that they have "no ***real*** solution", since the real number system has no number which represents the square root of –1. So once again, it becomes necessary to extend the number system, this time to the **Complex Numbers**, denoted $\mathbb{C}$ (which contains $\mathbb{R}$ and hence $\mathbb{Q}$ and hence $\mathbb{Z}$ and hence $\mathbb{N}$ - so making $\mathbb{N}$ the 'simplest' number set and $\mathbb{C}$ the most 'complex'), where $\mathbb{C}$ contains a representation for solutions of the above problems. So what is the answer here?

To start with we need to know $\sqrt{-1}$. That's easy, just call it *j*.

So $\sqrt{-1} = j$

But what, I can hear you say, is *j*? Well, that's easy too. *j* is the square root of –1!

This *has* to be confusing. But if you cast your mind back to the time when you were learning negative numbers, you probably asked, “Yes, but what *is* –1? After all, you couldn’t count objects with negative numbers (that was the wrong visualisation) and similarly here, what do *j* pens look like? Again, this is the wrong visualisation (later!).

Revisiting the three quadratics above then, the solutions now become $\pm j$, $\pm 3j$ and $-2 \pm 3j$. These may not look much like solutions to you, but if they ‘fit’ the original equations, then they *must* be solutions. Problem solved!

Check out $x = 3j$ and $x = -3j$ back in the original equation $x^2 + 9 = 0$
***bearing in mind that if*** $j = \sqrt{-1}$ ***then, squaring both sides,*** $j^2 = -1$.

When $x = 3j$,
$x^2 + 9 = (3j)^2 + 9 = 3^2 \times j^2 + 9 = 9 \times (-1) + 9 = -9 + 9 = 0$, as required
So $x$ = 3$j$ satisfies the equation, so must be a solution.

When $x = -3j$,
$x^2 + 9 = (-3j)^2 + 9 = (-3)^2 \times j^2 + 9 = 9 \times (-1) + 9 = -9 + 9 = 0$, as required.
So $x$ = -3$j$ satisfies the equation, so must be the other solution.

Let’s now see if $x$ = -2 + 3$j$ is a solution of $x^2 + 4x + 13 = 0$.

Substitute for $x$ in the left hand side of the quadratic equation:
l.h.s. = $(-2 + 3j)^2 + 4(-2 + 3j) + 13$ = $(4 - 6j - 6j + 9j^2) + 4(-2 + 3j) + 13$

Note: $(-2 + 3j)^2$ multiplies out in the same way as you would expand $(x + 2)^2$, for example.

but $j^2$ = -1, so
l.h.s. = $(-5 - 12j) + 4(-2 + 3j) + 13 = -5 - 12j - 8 + 12j + 13 = 0 =$ r.h.s., as required.
So $x$ = -2 + 3$j$ satisfies the equation, so must be a solution.

-2 + 3$j$ is called a complex number.

A ***general* complex number** is usually denoted by the letter, $z$ (zed) and written

***z = a +bj***
where $a$ is called the **real part** and $b$ is called the **imaginary part**

This can be written as Re{$z$} = $a$ and Im{z} = $b$
(note that when you give the imaginary part, the $j$ is not included)

The use of the words ‘imaginary’ and ‘complex’ are unfortunate since ‘imaginary’ can mean ‘doesn’t really exist’ and ‘complex’ can mean ‘difficult’. However, these numbers are no more imaginary and no more complex (as in complicated), say, than negative numbers. ‘Imaginary’ here is used in the context, ‘this part of the solution is not the *real* part’ and ‘complex’ in the context that the number is a *composite* of two different types of numbers (i.e. real and imaginary).

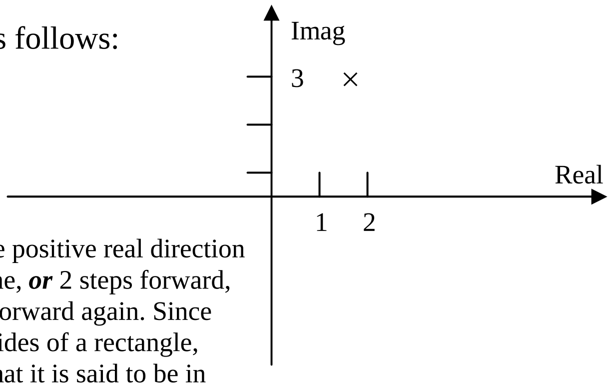
## Visualising Complex Numbers

The use of directed numbers on the real number line was useful for interpreting negative numbers. Plus 3 (+3) could be represented by 3 steps forward in the positive direction; -3 could represent 3 steps forward in the negative (opposite) direction (or 3 steps backwards in the positive direction!).

Does this help us visualising complex numbers? Partly!

The visualisation generally used is the ***Argand Diagram***. Monsieur Argand decided that since a complex number contains a real part, the use of the real number line could be retained, but he then extended this idea so that the imaginary part would also be represented by a number line, but at right angles to the real number line. In this way, a complex number exists in a 2-dimensional plane, unlike a real number, which exists only along a 1-dimensional line.

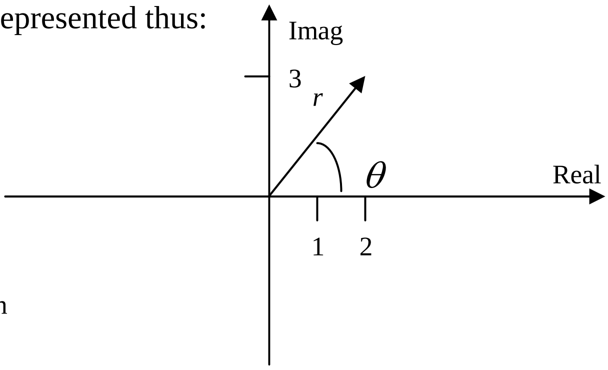
It is usual to set up an Argand Diagram as follows:




So a complex number such as $2 + 3j$ exists at a point +2 in the 'real' direction and +3 in the imaginary direction.

This is equivalent to taking 2 steps forward in the positive real direction and then 3 steps to the left off the real number line, ***or*** 2 steps forward, turn through $90^{\circ}$ anticlockwise and then 3 steps forward again. Since this is equivalent to getting to the point via two sides of a rectangle, when a complex number is written in $a + bj$ format it is said to be in ***Rectangular Form*** (sometimes known as ***Cartesian Form***).

Rectangular form is not the only way to represent a complex number on an Argand Diagram. Instead of merely a point, a complex number can be represented by a directed line segment (arrow) *from* the origin of the Argand Diagram *to* the point itself. So $2 + 3j$ can also be represented thus:




The line segment has a length $r$ and a direction $\theta$ anticlockwise from the positive real axis. Since this is equivalent to getting to the point *directly* from a central position, or 'pole', when a complex number is expressed in terms of $r$ and $\theta$ it is said to be in ***Polar Form***.

In polar form a complex number can also be considered as being equivalent to a ***vector*** or a ***phasor***, since both a complex number and a vector (or phasor) have a magnitude and a direction. In the case of the complex number, these are usually denoted by the length, $r$, and the angle (to the positive real axis), $\theta$, respectively.

One consequence of this equivalence is that complex numbers can be used to solve 2-dimensional vector and phasor problems.

Amongst other things, Theory Sheet 2 for Complex Numbers will explore the relationship between the two forms above and look at the basic rules of addition, subtraction, multiplication and division of complex numbers.