Name ……………………………………… Course ………………………………

Print and use this sheet in conjunction with ***MathinSite***'s 'Complex Numbers' applet.

Complex Numbers

Theory Sheet 2

Manipulation of Complex Numbers in Rectangular and Polar form

Learning Outcomes

- … have been outlined in Theory Sheet 1

Background

Theory Sheet 1 showed how the real number system, $\mathbb{R}$, was insufficient when trying to solve problems containing the square roots of negative numbers. It was necessary to extend the number system to incorporate 'imaginary' numbers and hence to develop the more comprehensive 'complex' number system. Although strange at first sight, the imaginary number, *j*, is no more or less imaginary than the number -1. Why ask, "What do *j* sheep look like?", when it is equally ridiculous to ask, "What do -1 sheep look like?" Counting objects, using either negative or imaginary numbers is an inappropriate visualisation. A good visualisation of complex numbers uses the Argand Diagram. This diagram is extremely useful in an engineering context, as we shall see, as it allows the use of complex numbers to solve a variety of engineering problems. However, first it is important to illustrate how complex numbers behave with respect to the basic operations of addition, subtraction, multiplication and division.

Before that, let us remind ourselves of where complex numbers arise and their general format. (See also Tutorial Sheet 1 in this series)

The solution of quadratic equations (of the form $ax^2 + bx + c = 0$) will have complex solutions if $b^2 < 4ac$. So for example, $x^2+4x+13=0$ gives, using the formula for solving a quadratic with $a = 1$, $b = 4$ and $c = 13$,

$$x=\frac{-b\pm\sqrt{b^2-4ac}}{2a}=\frac{-4\pm\sqrt{4^2-4\times1\times13}}{2\times1}=\frac{-4\pm\sqrt{-36}}{2}=\frac{-4\pm6j}{2}=-2\pm3j$$

The quadratic equation has two complex solutions, $x=-2+3j$ and $x=-2-3j$

The answers here are numbers of the form $a + bj$ (sometimes written $a + jb$) with real numbers a and b, and $a + bj$ is a <u>complex number</u>.

- a is called the <u>real part</u> of $a + bj$, and b is called the <u>**imaginary** part</u> of $a + bj$.
- If z denotes a complex number, the real part of z is sometimes written as Re z, and the imaginary part as Im z.
- If $a = 0$, the complex number is <u>wholly, or purely, imaginary</u>, e.g. $3j\ (\equiv 0+3j)$
- If $b = 0$, it is <u>wholly real</u>, e.g. $2\equiv 2+0j$ (so all real numbers are complex numbers with a zero imaginary part! i.e. the real number set is a subset of the complex number set)
- Two complex numbers are <u>equal</u> if and only if they have the same real and imaginary part.
- The complex number 0 is just $0 + j0$.

Manipulation of Complex Numbers in Rectangular form

Practically the first operations we performed with numbers in early schooldays were addition, subtraction, multiplication and division. As we were introduced to each new number system, we learned how to perform these operations. So again, with complex numbers, we need to be able to know how to add, subtract, multiply and divide.

It is safe to assume that the ordinary rules of arithmetic apply, so that the following rules for **addition, subtraction and multiplication** apply to any two complex numbers of the form $z_1 = a + bj$ and $z_2 = c + dj$:

Addition: $\mathbf{z_1 + z_2} = (a + bj) + (c + dj) = \mathbf{(a + c) + (b + d)j}$

Subtraction: $\mathbf{z_1 - z_2} = (a + bj) - (c + dj) = \mathbf{(a - c) + (b - d)j}$

Multiplication: $z_1 . z_2 = (a + bj).(c + dj) = ac + adj + bcj + bdj^2$

but remember $j^2 = -1$, so

$\mathbf{z_1 . z_2} = (a + bj)(c + dj) = \mathbf{(ac - bd) + (bc + ad)j}$

<u>*Note:*</u>

- In addition (or subtraction), add (or subtract) corresponding real parts and imaginary parts
- Multiplication is carried out in exactly the same way that you would multiply, for example, (x + 2)(2x – 3)
- don't try to remember these formulae, work each one out 'on the fly'.

As you become more adept at such calculations, it becomes possible to write down the answers directly. The first three examples that follow are fairly straightforward – just beware of "minus minuses"!

Examples of Complex Addition and Subtraction

1. $(2 + 3j) + (6 - j5) = 8 - 2j$ (remember $a + jb = a + bj$)
2. $(4 - 3j) - (6 - 2j) = -2 - j$
3. $\sqrt{2} + 5j - \left(6 - \sqrt{3}j\right) = \sqrt{2} - 6 + \left(5 + \sqrt{3}\right)j$

There's nothing wrong with leaving your answer in this square root format – although you can write it in decimals if you really want to.

Examples of Complex Multiplication

1. $(2 + 5j)(6 - 3j) = 12 - 6j + 30j - 15j^2$ $12 + 24j - 15(-1) = 27 + 24j$
2. $3j(4 - 6j) = 12j - 18j^2 = 18 + 12j$ (remember $j^2 = -1$)

Before looking at complex division it is necessary to first consider …

The Complex Conjugate of z, $\bar{z}$

If $z = a + bj$, then the complex conjugate, $\bar{z}$, is $\bar{z} = a - bj$

(i.e. just change the sign of the imaginary part)

Example: If $z = -2 - 3j$ then $\bar{z} = -2 + 3j$.

Note that these were the two roots of the quadratic equation on page 1 of this tutorial sheet. In fact, when a quadratic equation has complex solutions these always occur as complex conjugate pairs.

Obviously, complex conjugating twice returns us to the original number: $\bar{\bar{z}} = z$.

On its own, the complex conjugate doesn't appear useful, but it has one important property that allows us to perform complex division, that is, $z\bar{z}$ **is always wholly real**.

Proof: let z be any complex number $a + bj$ so that $\bar{z} = a - bj$, so

$$z\bar{z} = (a+bj)(a-bj) = a^2 - abj + abj - b^2j^2 = a^2 + b^2$$ **- wholly real**

Note that a complex number multiplied by its own complex conjugate is easily found from (real part)^2 + (imaginary part)^2

<u>Complex division</u> is always performed by *multiplying both numerator and denominator of the complex division by the complex conjugate of the numerator*. This will ensure a real denominator.

In general this will always look like:

If $(c + jd) \neq 0 + j0$ then $\dfrac{a+bj}{c+dj}$ can be calculated by writing

$$\frac{a+bj}{c+dj} \quad \text{as} \quad \frac{a+bj}{c+dj} \times \frac{c-dj}{c-dj}$$

$$= \frac{ac+bd}{c^2+d^2} + j\frac{bc-ad}{c^2+d^2} \equiv A + Bj$$

So when two complex numbers are divided, the result is a complex number.
Again, don't try to remember this formula, work each case out 'on the fly'.

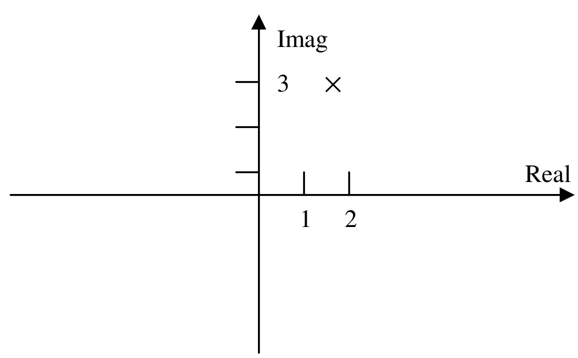
Example of Complex Division

$$\frac{3-5j}{-6+4j} = \frac{3-5j}{-6+4j} \times \frac{-6-4j}{-6-4j} = \frac{-38+18j}{36+16} = \frac{-38+18j}{52} = \frac{-19+9j}{26} = -\frac{19}{26} + \frac{9}{26}j$$

Complex Numbers in Polar Form

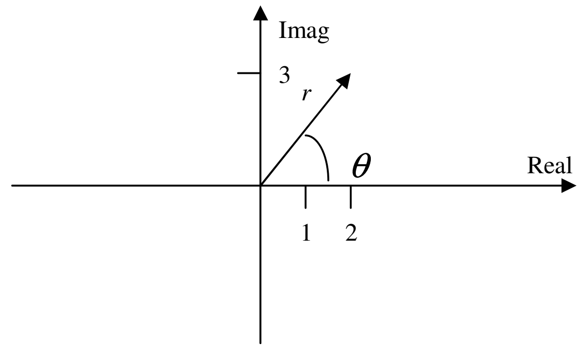
In Tutorial Sheet 1 we saw that complex numbers can be visualised in Rectangular Form (or Cartesian Form) as points in a 2-dimensional plane (an Argand Diagram), unlike real numbers, which exist only along a 1-dimensional line.

So a complex number such as 2 + 3*j* exists at a point reached by moving +2 in the 'real' direction and +3 in the imaginary direction.



Rectangular form is not the only way to represent a complex number on an Argand Diagram. Instead of merely a point, a complex number can be represented by a directed line segment (arrow) *from* the origin of the Argand Diagram *to* the point itself. So 2 + 3*j* can also be represented thus:

Note that this line segment can be thought of as a vector, or phasor, with a length r and a direction θ anticlockwise from the positive real axis. Since this is equivalent to getting to the point *directly* from origin, or 'pole', when a complex number is expressed in terms of r and θ it is said to be in ***Polar Form***.

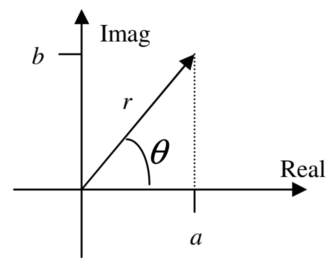


So, in the case of polar form, the complex number is specified by the length, r, and the angle (to the positive real axis), θ.

One consequence of being able to express complex numbers in polar form is that they can be used to solve 2-dimensional vector and phasor problems.

Conversion from Rectangular to Polar form and vice versa

Consider the following Argand Diagram in which the complex number $z = a + bj$ is illustrated with both its rectangular and polar 'coordinates' shown.



Dropping a vertical line from the end of the complex number to the real axis produces a right-angled triangle in which the non-hypotenuse sides are a and b and the hypotenuse is r. The angle measured from the positive real axis is denoted θ.

Applying trigonometrical ratios to this triangle gives

$$\cos\theta = \frac{a}{r} \text{ and } \sin\theta = \frac{b}{r} \quad \text{or} \quad \boxed{a = r\cos\theta \text{ and } b = r\sin\theta}$$

So, the complex number written in Rectangular Form, $z = a + bj$,

can be written in Polar Form as $z = r\cos\theta + jr\sin\theta$

(note in this case how it is more usual to write the j at the *beginning* of the imaginary term)

of course, with a common factor of r, this becomes

$z = r(\cos\theta + j\sin\theta)$. This is often abbreviated to $z = r\angle\theta$ or $z = \langle r, \theta \rangle$

Needless to say, it is possible to determine a complex number's Rectangular form from its polar form. Returning to the triangle above, Pythagoras' Theorem gives

$$r = \sqrt{a^2 + b^2} \text{ and trigonometry gives } \theta = \tan^{-1}\left(\frac{b}{a}\right)$$

NOTE: When using this latter expression, it is important to determine the correct quadrant for θ. To this end it is always advisable to sketch the Argand Diagram to ensure the correct angle is found.

In summary then

Rectangular Form: $z = a + bj$

Polar Form: $z = r(\cos\theta + j\sin\theta)$, or $z = r\angle\theta$ or $z = \langle r, \theta \rangle$

Rectangular to Polar Conversion: $a = r\cos\theta$ and $b = r\sin\theta$

Polar to Rectangular Conversion: $r = \sqrt{a^2 + b^2}$, $\theta = \tan^{-1}\left(\frac{b}{a}\right)$

NOTE: Many scientific calculators include rectangular to polar conversion, and vice versa, as standard. If your calculator has keys labelled "R → P", or "→ (r, θ)", or equivalent, read your calculator manual to see how these keys work. It will be well worth your while, since these return any angles in the correct quadrant – automatically!

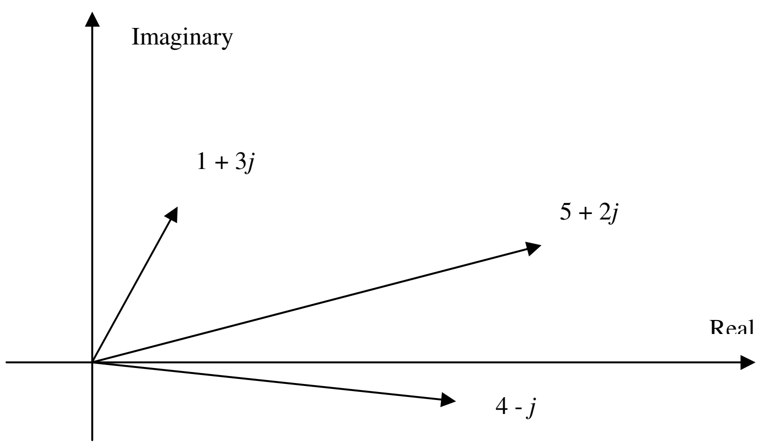
In Polar Form a complex number has a ***magnitude***, r, and a ***direction***, θ. These are properties shared with vectors and phasors. This means that complex numbers can be used to solve vector, or phasor, problems and can be illustrated in the following.

Addition of Complex Numbers on an Argand Diagram
(… and its application to "Parallelogram of Forces" problems)

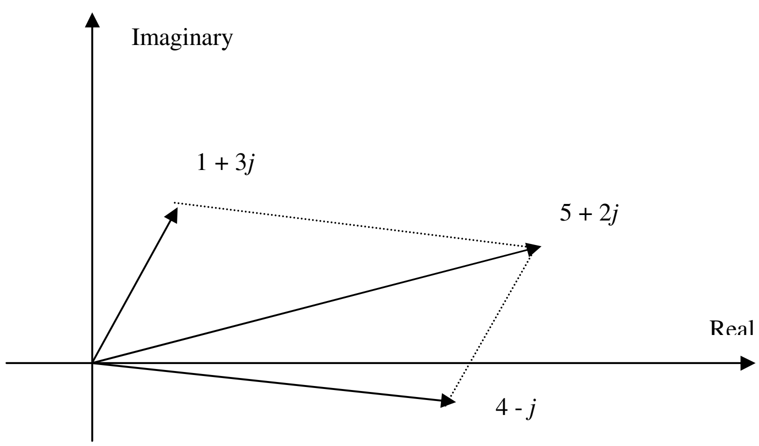
Consider adding the two complex numbers $z_1 = 1 + 3j$ and $z_2 = 4 - j$. The answer here is, of course, easily determined by adding the real parts and the imaginary parts separately.

So $$z_1 + z_2 = (1 + 4) + (3 - 1)j = 5 + 2j.$$

The Argand Diagram visualisation of this is:



Now notice that, by joining the ends of the original complex numbers to the end of the resultant complex number, the following diagram is obtained:



Some of you will recognise this as the "Parallelogram of Forces" diagram. Here, the forces (or vectors, or complex numbers) z_1 and z_2 combine to give a ***resultant*** $5 + 2j$.

In a vector/force/phasor, problem, the vector will not usually be specified in Rectangular Form. In the case above the complex number, or force/vector/phasor, z_1 will normally be expressed with a magnitude and a direction, i.e. in polar form.

So here	$z_1 = 1 + 3j$	$= 3.1623\angle 71.565^o$	(5 s.f.)
and	$z_2 = 4 - j$	$= 4.1231\angle -14.036^o$	(5 s.f.)
The resultant is	$5 + 2j$	$= 5.3852\angle 21.801^o$	(5 s.f.)

In words this says that there are two vectors, one with a magnitude 3.1623 making an angle of 71.565^o with some datum axis (usually the positive real axis), the second with a magnitude 4.1231 and angle -14.036^o which, when added, result in a vector with magnitude 5.3852 and angle 21.801^o.

A most important point to note here is that ***it is not possible to add complex numbers*** (or phasors or vectors) ***in polar form*** (3.1623 + 4.1231 is NOT equal to 5.3852 and $71.565^o + (-14.036^o)$ is NOT equal to 21.801^o). Here, as in all such cases, the addition has to be done in rectangular form.

Example A vector has a magnitude of 10 units and makes an angle of 35^o to a datum axis. A second vector has a magnitude of 20 units and makes an angle of 120^o to the same axis. Find their resultant.

A problem like this is probably best tabulated. (The datum axis described is quite arbitrary, so the positive real axis is automatically chosen)

Step 1: polar to rectangular conversion →

Vector	Polar Form	Rectangular Form
z_1 (Vector 1)	$10\angle 35^o$	$8.1915 + 5.7358j$
z_2 (Vector 2)	$20\angle 120^o$	$-10 + 17.3205j$
$z_1 + z_2$ (Resultant)	$23.1271\angle 94.48^o$	$-1.8085 + 23.0563j$

Step 2: add vectors in rectangular form

← Step 3: convert back to polar form

So the resultant vector has a magnitude of 23.1271 and makes an angle of 94.48^o with the datum axis.

Always sketch an Argand Diagram when solving such problems. A reasonable sketch should at least give an indication that the resultant 'looks about right'! Try it now with this problem. You can, of course, check this result using the accompanying *MathinSite* 'Complex Numbers' applet available from

http://mathinsite.bmth.ac.uk/html/applets.html

using the 'Polar Addition' option from the drop-down menu in the top right of the applet window.

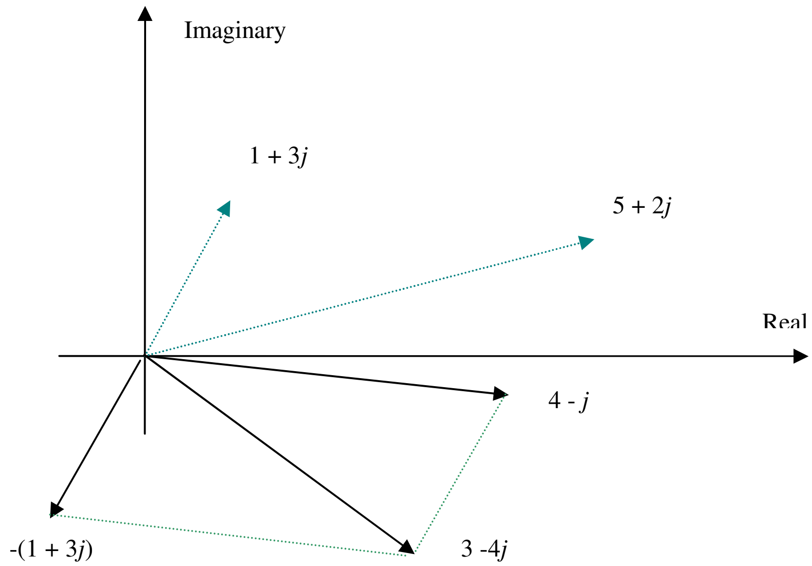
Needless to say, this process can be extended to three, or indeed *any* number, of vectors.

Subtraction of Complex Numbers on an Argand Diagram

Consider again the two complex numbers $z_1 = 1 + 3j$ and $z_2 = 4 - j$. To subtract z_1 from z_2, simply subtract the real parts and the imaginary parts separately.

So $\quad z_2 - z_1 = (4 - 1) + (-1 - 3)j = 3 - 4j$.

The Argand Diagram visualisation of this is:



First notice that ***subtracting*** z_1 ***is exactly the same as adding*** $-z_1$. In which case, the "Parallelogram of Forces" strategy can be used again on the Argand Diagram, but this time by adding vectors z_2 and MINUS z_1.

Here	$-z_1$	$= -(1 + 3j)$	$= 3.1623\angle -108.435^o$
and	z_2	$= 4 - j$	$= 4.1231\angle -14.036^o$
resultant is	$z_2 - z_1 = z_2 + (-z_1)$	$= 3 - 4j$	$= 5\angle -53.130^o$

Note that the required angle for $-z_1$ is $-(180 - 71.565)^o = -108.435^o$

Note also that, as with addition, ***it is not possible to subtract complex numbers*** (phasors /vectors) ***in polar form*** (3.1623 + 4.1231 is NOT equal to 5 and $-108.435^o + (-14.036^o)$ is NOT equal to -53.130^o). Here again, any subtraction has to be done in rectangular form.

Exercises (It is possible to corroborate certain answers using the accompanying "Complex Numbers" applet.)

1. Convert the following complex numbers into Polar Form, ensuring that the angle obtained is in the correct quadrant.
   (a) $3 + 4j$, (b) $-3 + 5j$, (c) $-6 - 7j$, (d) $5 - 12j$

2. Convert the following complex numbers into Rectangular Form.
   (a) $5\angle 30^0$, (b) $15\angle -30^0$, (c) $3\angle 230^0$, (d) $3\angle -130^0$

3. The answers you obtained in Q 2 (c) and (d) are the same. Explain why.

4. If $z_1 = 3 - 5j$, $z_2 = 5 + 6j$ and $z_3 = -2 + 5j$, find, in Rectangular form, the values of
   a. $z_1 + z_2$
   b. $z_1 + z_3$
   c. $z_1 - 4z_2 + 5z_3$
   d. $z_1 \times z_3$

e. $\dfrac{z_1 \times z_3}{z_2}$

5. Convert all your answers in Q. 4 into Polar Form.

6. Given $z_1 = 6\angle 50^0$, $z_2 = 3\angle -50^0$ and $z_3 = 5\angle 120^0$, find, in Polar Form z_1 - 4z_2 +5z_3.

7. Three forces have magnitudes 3N (newton), 6N and 15N each making an angle to a datum axis of 30^o, 60^o and -240^o respectively. Find the resultant of these three forces and sketch this system of forces on an Argand Diagram.

8. Show that the real and imaginary parts of $z = \dfrac{2+3j}{1+2j} - \dfrac{8+j}{6-j}$ are $\dfrac{61}{185}$ and $-\dfrac{107}{185}$.