Complex Numbers

Theory Sheet 3

Multiplication and Division of Complex Numbers in Polar form. De Moivre's Theorem

Learning Outcomes

- … have been outlined in Theory Sheet 1

Review

Theory Sheets 1 and 2 showed how complex numbers can be

- obtained when solving quadratic equations
- added, subtracted, multiplied and divided in Rectangular ($a + bj$) Form
- converted into Polar ($r\angle\theta$) Form, and back
- visualised using the Argand Diagram (inc. operations on complex numbers)
- "added and subtracted in Polar Form"
- used to solve engineering problems relating to vectors and phasors.

Addition/Subtraction of Complex Numbers in Polar Form - a Review

REMEMBER: It is not directly possible to add or subtract complex numbers in Polar Form. The vector example on p. 6 of Complex Numbers, Theory Sheet 2, shows that it is necessary to convert from Polar to Rectangular Form first, perform the addition (and/or subtraction) and then convert back to Polar Form. As a reminder …

Example 1.

$$\begin{aligned}&3\angle 30^0 + 5\angle -80^0\\&=(2.5981+1.5j)+(0.8682-4.9240j)\\&=3.4663-3.4240j\\&=4.8723\angle -44.65^0\end{aligned}$$

This can be verified using Polar Addition in the accompanying MathinSite applet

Example 2.

$$\begin{aligned}&8\angle 130^0 + 2\angle -70^0 - 4\angle 50^0\\&=(-5.1423+6.1284j)+(0.6840-1.8794j)-(2.5712+3.0642j)\\&=-7.0295+1.1848j\\&=7.1286\angle 170.43^0\end{aligned}$$

Multiplication of Complex Numbers in Polar form

Are there simple methods for multiplication/division in Polar Form? There is good news here; there are! The bad news is; we need to undertake a bit of general analysis to determine these methods. This needs the trigonometrical identities:

$$\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2 = \cos(\theta_1+\theta_2) \qquad \cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 = \cos(\theta_1-\theta_2)$$
$$\sin\theta_1\cos\theta_2 + \cos\theta_1\sin\theta_2 = \sin(\theta_1+\theta_2) \qquad \sin\theta_1\cos\theta_2 - \cos\theta_1\sin\theta_2 = \sin(\theta_1-\theta_2)$$
$$\sin^2\theta + \cos^2\theta = 1$$

Consider two complex numbers in Polar Form, z_1 and z_2, where $z_1 = r_1\angle\theta_1$ and $z_2 = r_2\angle\theta_2$. Remember that this notation is a shortened form for $z_1 = r_1\left(\cos\theta_1 + j\sin\theta_1\right)$ and $z_2 = r_2\left(\cos\theta_2 + j\sin\theta_2\right)$. So multiplying z_1 and z_2,

$$\begin{aligned}
z_1 \times z_2 &= r_1\angle\theta_1 \times r_2\angle\theta_2 \\
&= r_1\left(\cos\theta_1 + j\sin\theta_1\right)\times r_2\left(\cos\theta_2 + j\sin\theta_2\right) \\
&= r_1 \times r_2 \times\left(\cos\theta_1 + j\sin\theta_1\right)\times\left(\cos\theta_2 + j\sin\theta_2\right) \\
&= r_1 r_2 \times\left(\cos\theta_1 \times\cos\theta_2 + \cos\theta_1 \times j\sin\theta_2 + j\sin\theta_1\times\cos\theta_2 + j\sin\theta_1\times j\sin\theta_2\right) \\
&= r_1 r_2\left(\left(\cos\theta_1\cos\theta_2 - \sin\theta_1\sin\theta_2\right) + j\left(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2\right)\right) \\
&= r_1 r_2\left(\cos\left(\theta_1+\theta_2\right) + j\sin\left(\theta_1+\theta_2\right)\right) \\
&= r_1 r_2\angle\left(\theta_1+\theta_2\right)
\end{aligned}$$

This really amazing – and helpful – result says:

$$r_1\angle\theta_1 \times r_2\angle\theta_2 = r_1 r_2\angle\left(\theta_1+\theta_2\right)$$

or

When multiplying complex numbers in Polar Form,
simply multiply the magnitudes and add the angles.

Examples of Complex Multiplication in Polar Form

1. $4\angle 30^0 \times 5\angle 85^0 = 20\angle 115^0$
2. $14\angle 30^0 \times 3\angle 160^0 = 42\angle 190^0$ OR $42\angle -170^0$
3. $3.9\angle 30^0 \times 2.3\angle -60^0 = 8.97\angle -30^0$
4. $3.9\angle\frac{\pi}{3} \times 2.3\angle\frac{2\pi}{5} = 8.97\angle\frac{11\pi}{15}$
5. $2\angle\frac{2\pi}{3} \times 3\angle\frac{6\pi}{7} = 6\angle\frac{32\pi}{21}$ *OR* $6\angle -\frac{10\pi}{21}$

NOTE: In Polar Form angles can be quoted in the range $0^0 \le \theta < 360^0$, but can equally be given in the ranges $0^0 \le \theta \le 180^0$ and $-180^0 < \theta \le 0^0$.
So that in Example 2 above, $42\angle 190^0$ OR $42\angle -170^0$ are equally valid answers.

Division of Complex Numbers in Polar form

There is a simple method for division of complex numbers in Polar Form as well. The analysis follows a similar pattern to that for multiplication.

Consider two complex numbers in Polar Form, z_1 and z_2, where $z_1 = r_1\left(\cos\theta_1 + j\sin\theta_1\right)$ and $z_2 = r_2\left(\cos\theta_2 + j\sin\theta_2\right)$.

$$\frac{z_1}{z_2} = \frac{r_1\angle\theta}{r_2\angle\theta_{2\,1}} = \frac{r_1\left(\cos\theta_1 + j\sin\theta_1\right)}{r_2\left(\cos\theta_2 + j\sin\theta_2\right)} = \frac{r_1\left(\cos\theta_1 + j\sin\theta_1\right)}{r_2\left(\cos\theta_2 + j\sin\theta_2\right)}\times\frac{\left(\cos\theta_2 - j\sin\theta_2\right)}{\left(\cos\theta_2 - j\sin\theta_2\right)}$$

$$= \frac{r_1}{r_2} \times \frac{\left(\cos\theta_1 \times \cos\theta_2 - \cos\theta_1 \times j\sin\theta_2 + j\sin\theta_1 \times \cos\theta_2 - j\sin\theta_1 \times j\sin\theta_2\right)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2} \times \frac{\cos\theta_1\cos\theta_2 + \sin\theta_1\sin\theta_2 + j\left(\cos\theta_1\sin\theta_2 + \sin\theta_1\cos\theta_2\right)}{\cos^2\theta_2 + \sin^2\theta_2}$$

$$= \frac{r_1}{r_2}\left(\cos\left(\theta_1 - \theta_2\right) + j\sin\left(\theta_1 - \theta_2\right)\right)$$

$$= r_1 r_2 \angle\left(\theta_1 - \theta_2\right)$$

Again, an amazing – and helpful – result that says:

$$\frac{r_1\angle\theta_1}{r_2\angle\theta_2} = r_1 r_2 \angle\left(\theta_1 - \theta_2\right)$$

or

When dividing complex numbers in Polar Form,
simply divide the magnitudes and subtract the angles.

Examples of Complex Division in Polar Form

- $\dfrac{4\angle 30^0}{5\angle 85^0} = 0.8\angle -55^0$
- $\dfrac{14\angle 30^0}{2\angle 160^0} \times = 7\angle -130^0 \quad \left(\text{OR} \quad 42\angle 230^0\right)$
- $\dfrac{3.9\angle 30^0}{1.3\angle -60^0} = 3\angle 90^0$
- $\dfrac{3.9\angle \frac{\pi}{3}}{1.3\angle \frac{2\pi}{5}} = 3\angle\left(\frac{\pi}{3} - \frac{2\pi}{5}\right) = 3\angle\left(-\frac{\pi}{15}\right)$
- $2\angle\frac{2\pi}{3} \times 10\angle\frac{6\pi}{7} \div 5\angle\frac{3\pi}{5} = \dfrac{2\times 10}{5}\angle\left(\frac{2\pi}{3} + \frac{6\pi}{7} - \frac{3\pi}{5}\right) = 4\angle\frac{97\pi}{105}$

Powers and Roots of Complex Numbers

Powers

Using the rule for multiplication of Complex numbers in Polar Form, it is easy to see that for a complex number, $z = r\angle\theta$,

$z^2 = r\angle\theta \times r\angle\theta = rr\angle\left(\theta + \theta\right) = r^2\angle 2\theta$ and

$z^3 = r\angle\theta \times r\angle\theta \times r\angle\theta = rrr\angle\left(\theta + \theta + \theta\right) = r^3\angle 3\theta$ and so on.

We can deduce that, in general, $z^n = r^n\angle n\theta$, and this is de Moivre's Theorem (in its simpler form).

i.e. de Moivre's theorem is $\left(r\angle\theta\right)^n = r^n\angle n\theta$

It is very simple to apply:

- $\left(4\angle 30^0\right)^3 = 64\angle 90^0$
- $\left(2\angle 50^0\right)^{10} = 1024\angle 500^0 \equiv 1024\angle 140^0$

(find the remainder when multiples of 360^0 have been removed)

- $\left(3\angle 140^0\right)^5 = 243\angle 700^0 \equiv 243\angle 340^0 \quad OR \quad 243\angle -20^0$

This is particularly useful if the complex number is given in Rectangular Form,

- $\left(3+4j\right)^{10} = \left(5\angle 53.130^0\right)^{10} = 9765625\angle 531.30^0 = -9653260.614 + 1477156.448j$

This uses 5 significant figure accuracy for the angle 53.130°
So note how silly my answer is, I'm using 5 s.f. input and writing the answer to 10 s.f!

Here, there is some good news – and some bad news.
Good News: It is much, much quicker to convert to Polar Form, apply de Moivre's Theorem and then convert back than to multiply the (3 + 4*j*) by itself 10 times over!
Bad News: Unfortunately the calculations, including the R → P and then P → R conversions, introduce rounding errors.

Even using ***all*** the digits allowable on *my* calculator, which displays 10 significant figures, I obtain -9653287.001 + 1476983.999*j*. Compare with the previous answer.

Bearing in mind that the original problem, $\left(3+4j\right)^{10}$, requires *whole-number* answers, the answer here would appear to be -9653287 + 1476984*j*.

So BEWARE: using de Moivre's Theorem to obtain simplifications to problems such as $\left(3+4j\right)^{10}$ is quick, but can introduce inaccuracies.

Roots

De Moivre's Theorem can also be found for finding the roots (e.g. square roots, cube roots, etc) of complex numbers. However, the Theorem itself, which appeared in its simpler form above, needs to be extended to accommodate the fact that there are TWO square roots (e.g. the square root of 9 is +3 *OR* -3), THREE cube roots, FOUR fourth roots, and so on – the above de Moivre formula would only ever give ONE root.

Bear in mind that an angle of, say,	30°	on an Argand Diagram	
is the 'same' angle as	30° + 360°		= 390°,
which is the same as	30° + 720°	= 30° + 2 x 360°	= 750°,
which in turn, is the same as	30° + 1080°	= 30° + 3 x 360°	= 1110°, etc
which, in general, is the same as	30° + *k* x 360°		

So any angle, θ, can be represented on an Argand Diagram by $\theta^{\text{o}} + 360k^{\text{o}}$
for any *k* = …-3, -2, -1, 0, 1, 2, 3, 4,…

So the more comprehensive version of de Moivre's theorem (*only used for finding roots*) is

$$\left(r\angle\theta^0\right)^n = r^n\angle n\left(\theta^0 + 360k^0\right)$$
where *k* = 0, 1, 2, 3, 4,…

Obviously, this could also be used with radian measure, in which case this would be

$$\left(r\angle\theta\right)^n = r^n\angle n\left(\theta + 2\pi k\right)$$
where *k* = 0, 1, 2, 3, 4,…

This form is also straightforward to apply

- $\sqrt{3+4j} = (3+4j)^{1/2} = \left(5\angle\left(53.130^0 + 360k^0\right)\right)^{1/2} = 5^{1/2}\angle\left(26.565^0 + 180k^0\right)$
    - for $k = 0$,
      $\sqrt{3+4j} = 5^{1/2}\angle\left(26.565^0\right) = 2.2361\angle 26.565^0 = 2.0000 + 1.0000j$
    - for $k = 1$,
      $\sqrt{3+4j} = 5^{1/2}\angle\left(206.565^0\right) = 2.2361\angle 206.565^0 = -2.0000 - 1.0000j$

NOTES:

- As expected, just as $\sqrt{9} = \pm 3$, so here $\sqrt{3+4j} = \pm\left(2.0000 + 1.0000j\right)$
- *Square roots* are separated on an Argand diagram by 180^o ($360^o/2$)
- Working and answers here are given to 5 s.f. The answers are, for this problem, *exactly* $\pm(2+j)$, but otherwise, once again, watch out for rounding errors.
- There are only TWO square roots, so once both have been found there is no point looking for more (by using k = 2, 3, 4, etc since these will only repeat those already found).

- Find the cube roots of 8.

  An obvious cube root here is 2 since $2^3 = 8$. But where are the others? The reason they are not easy to guess is because the other two roots are *complex*!

  $\sqrt[3]{8} = \sqrt[3]{8+0j} = \left(8\angle\left(0+2\pi k\right)\right)^{1/3}$ (let's use radians this time)

  So $\sqrt[3]{8} = 8^{1/3}\angle\frac{1}{3}\times\left(0+2\pi k\right) = 2\angle\frac{2\pi}{3}k$

    - For $k = 0$, $\sqrt[3]{8} = 2\angle 0 = 2 + 0j = 2$ (the 'obvious' answer)
    - For $k = 1$, $\sqrt[3]{8} = 2\angle\frac{2\pi}{3} = -1 + 1.7321j$
    - For $k = 2$, $\sqrt[3]{8} = 2\angle\frac{4\pi}{3} = -1 - 1.7321j$

NOTES:

- *Cube roots* are separated on an Argand diagram by $\frac{2\pi}{3} \equiv 120^0\left(= 360^0/3\right)$.
- *Fourth roots* will be separated by 360/4 = 90^o, *fifth roots* by 360/5 = 72^o, etc
- There are only THREE roots, so once all three have been found there is no point looking for more (by using k = 3, 4, etc).
- Since the problem is set in Rectangular Form (8 + 0*j*), the answer should be in Rectangular Form also. Use this as a general rule: Start format = End format.

You can, of course, check this result using the accompanying *MathinSite* 'Complex Numbers' applet available from
http://mathinsite.bmth.ac.uk/html/applets.html
using the 'Rectangular Roots'/'3 roots' option from the drop-down menus in the top right of the applet window. 'Snap to grid' is useful to obtain 8 + 0*j* exactly.

Exercises

Use the ***MathinSite*** "Complex Numbers" applet to corroborate your answers where possible.

1. Find
   (a) $5\angle 30^0 + 7\angle -30^0$ (b) $3\angle 230^0 + 3\angle -130^0$,

2. Find
   (a) $5\angle 30^0 \times 7\angle -30^0$ (b) $5\angle 30^0 \div 7\angle -30^0$
   (c) $3\angle 230^0 \times 3\angle -130^0$ (d) $3\angle 230^0 \div 3\angle -130^0$
   (e) $\dfrac{3\angle 230^0 \times 7\angle -30^0}{5\angle 30^0 \times 3\angle -130^0}$

3. Find $(3 - j)^5$ by
   (a) multiplying out the brackets in Rectangular Form
   (b) converting to Polar Form (with 4 significant figure accuracy), using de Moivre's Theorem and converting back to Rectangular Form.
   Compare your answers for (a) and (b) and account for any differences.

4. Find
   (c) The fourth roots of 2
   (d) The fifth roots of $-3 + j$
   (e) The cube roots of $1 + 2j$
   (f) The square roots of $3\text{-}4j$
   Sketch each case on an Argand Diagram noting the 'angles of separation' of the roots.

5. Find
   (a) $(3.234 - 1.675j)^{2/3}$
   (b) $\sqrt[4]{(2-3j)^3}$