Differentiation 1 – From Chords to Tangents
Some Theory

Learning Outcomes

After using the applet and accompanying tutorial and theory sheets you should

- be aware of the difference between a chord and a tangent
- be able to handle the different notation used for the gradient of a chord and of a tangent
- have developed, through experimentation, an understanding of basic differentiation and how chords and tangents are related to the differentiation process
- be able to answer "what if ...?" questions about basic differentiation.

It is recommended that you use the associated applet and complete the accompanying Software Tutorial sheet before reading this Theory sheet.

Now that you have a feel for the difference between chords and tangents, this is a good place to finish with the applet and introduce some important notation and theory.

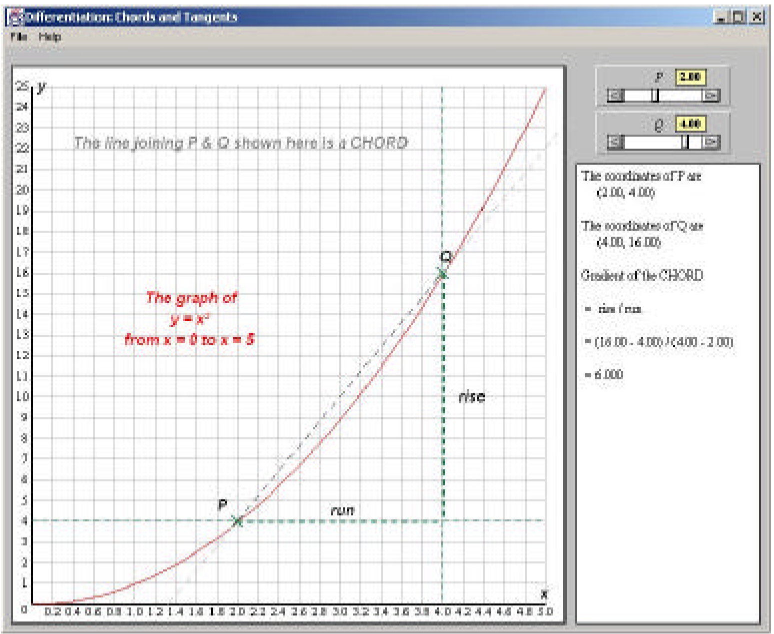
Note that the generalisation from $y = x^2$ to other, more complicated, functions will be left for future development on the '***MathinSite***' web site.

The accompanying applet and tutorial sheet uses the curve $y = x^2$ (only) to illustrate how the gradients of chords and tangents can be evaluated and shows the difference between them. In differential calculus, *the gradient of the tangent is much more important than the gradient of the chord*. However, we have to use the gradient of the chord to eventually find the gradient of the tangent.

Gradient of the Chord

The **chord** through PQ has gradient = $\dfrac{\text{rise}}{\text{run}}$.

Here the 'rise' is the difference between two y values. The symbol dy is used for this difference in y values. The corresponding 'run' values will be the difference of two x-values and is denoted dx .



So the gradient of the chord will be given by

$$\text{gradient of the chord} = \frac{\text{rise}}{\text{run}} = \frac{y_2 - y_1}{x_2 - x_1} = \frac{dy}{dx}$$

Since we are interested in what happens as the chord becomes the tangent, the points P and Q will be moved 'closer' together and, ultimately, become coincident. Thus dx and dy will usually be 'small' values (…although 'small' is a relative term).

Now the chord between P and Q becomes the tangent at P when Q moves until it is coincident with P. "As Q tends towards P" is written mathematically as "Q → P", but note that Q → P is equivalent to saying $dx \to 0$ (i.e. the difference between P and Q's x-values becomes zero).

The gradient of the chord becomes the gradient of the tangent as $dx \to 0$. It is necessary to distinguish between the gradient of the chord and the gradient of the tangent, so while the gradient of the chord is denoted $\frac{dy}{dx}$, ***the gradient of the tangent will be denoted*** $\frac{dy}{dx}$.

Mathematically then, the limit of $\frac{dy}{dx}$ as $dx \to 0$ is $\frac{dy}{dx}$, or more concisely written,

$$\boxed{\lim_{dx\to 0}\left\{\frac{dy}{dx}\right\} = \frac{dy}{dx}}$$

Now we're ready to see why the gradient of the tangent on $y = x^2$ is twice the x value.

The gradient of the tangent to $y = x^2$.

Take point P as the general point P(x, y) on $y = x^2$ and put Q a 'small' distance dx away on the curve such that its co-ordinates are Q($x + dx$, $y + dy$). Since point Q is on the curve, its co-ordinates must satisfy the curve's equation, so instead of $y = x^2$ we can now write

$$y + dy = (x + dx)^2$$

Expanding the brackets on the right hand side gives

$$y + dy = x^2 + 2x\,dx + dx^2$$

but, since $y = x^2$, the y on the left and the x^2 on the right cancel to give

$$dy = 2x\,dx + dx^2$$

and dividing both sides by dx results in

$$\frac{dy}{dx} = 2x + dx$$

and this is the ***gradient of the chord*** between P and Q.

Of more interest here, though, is the gradient of the tangent, found by letting $dx \to 0$ (on both sides of this equation).

So $\lim_{dx\to 0}\left\{\frac{dy}{dx}\right\} = \lim_{dx\to 0}\{2x + dx\}$ OR $\frac{dy}{dx} = 2x$ (since $dx \to 0$)

... and this corroborates the information found using the applet, i.e. the gradient of the tangent at any point on $y = x^2$ is given by twice the x value.

In your Calculus courses you will come across the differentials of other functions such as exponential and trigonometric. Having used this applet and its accompanying tutorial and theory sheets, you should now be aware of, at least, the meaning of the term differentiation and its associated notation and have some knowledge of the difference between chords and tangents and their gradients.