

## Differentiation 2: Differentiation of some Basic Functions

### The Software Tutorial 1: $y = x^2$



#### Learning Outcomes

After using the applet and *this* tutorial sheet you should

- recognise that the gradient of a curve can, itself, be represented and plotted as a curve
- recognise first and second order derivatives of basic functions
- be able to use the different notation used for first and second order derivatives
- understand the concept of a 'rate of change'
- develop, through experimentation, a deeper insight into the differentiation of  $y = x^2$
- be able to answer "what if ...?" questions about the 1<sup>st</sup> and 2<sup>nd</sup> differentials of  $y = x^2$ .

#### Introduction

Before using this applet and its associated worksheets you should have used the applet, "Differentiation 1: From Chords to Tangents" - also available from the *MathinSite* web site. From there you will have seen how the gradient of the chord between two points P and Q becomes the gradient of the tangent at the point P (or Q) as the two points coincide. You will also have seen and used the "dy by dx" notation for differentiation.

The accompanying applet is used to introduce further fundamental concepts of differentiation and, consequently, uses only basic functions to put across these ideas.

#### Loading the Software

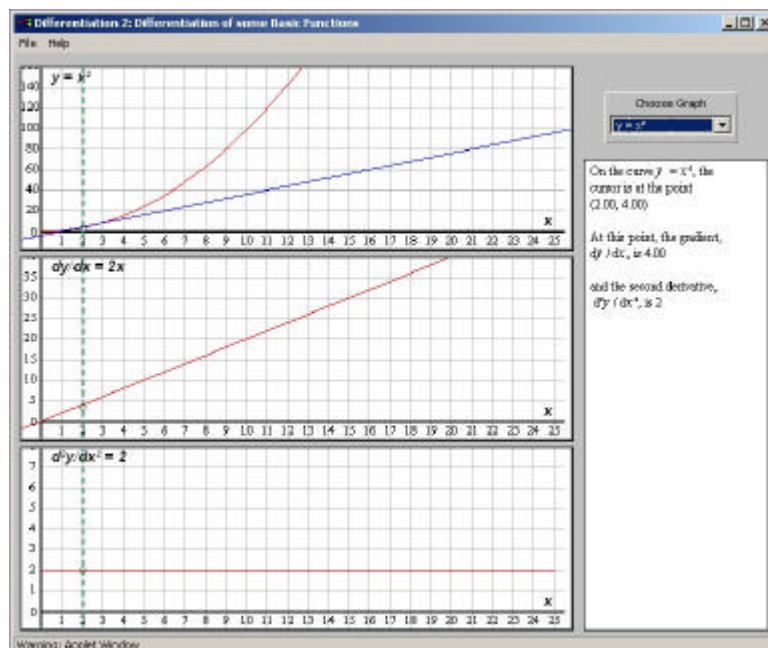
Select "Differentiation 2" from the web page from which this tutorial sheet was obtained. When loaded, you will see the "Differentiation 2: Differentiation of some Basic Functions" applet window similar to that shown below.

#### Running the applet

The applet shows three graphics windows, from top to bottom, (a) the function, (b) the first derivative of the function and (c) the second derivative.

Superimposed on all three is a (dashed) vertical cursor line which can be 'grabbed' by placing the mouse cursor on, or near, the line in ANY of the graphics areas and then a click / hold / drag operation will move the cursor line on all three graphs.

The choice of function is obtained from the drop-down box on the upper right of the applet window.



## The Tutorial

When loaded, the applet shows the graph  $y = x^2$  in the top window with the tangent to the curve (the blue line) at the point (2, 4) superimposed. The second graphics window shows the graph of the derivative, namely  $dy/dx = 2x$  and the third shows the second derivative graph, namely  $d^2y/dx^2 = 2$ . Superimposed on all three is the green dotted cursor line showing on all three graphs that the cursor is at  $x = 2$ . The text area on the right shows the value of the function and its derivatives at the cursor position (it is not always possible to read these values accurately enough from the graphs).

### The graph $y = x^2$

'Grab' the cursor in the top graphics window and slowly move it to the right as far as  $x = 8$ . Notice how the blue tangent line moves along with the cursor – and how the *gradient*, or slope, of the tangent line changes.

In fact, as  $x$  increases, the gradient of the tangent to  $y = x^2$  also .....

**Remember that  $dy/dx$  is the gradient of the tangent.**

This increase in  $dy/dx$  as  $x$  increases can be seen by completing the following table.

Cursor at $x =$	Cursor at $y =$	$\frac{dy}{dx} =$	$\frac{d^2y}{dx^2} =$
2			
4			
6			
8			
10			

First grab the cursor line and move it back to  $x = 2.00$  (this can take some practice and a steady hand!). Read off the values for the  $y$ -coordinate and the values of the first and second differentials from the text area.

Note that as you reach  $x = 10$ , the top graph begins to scroll so that higher points on  $y = x^2$  can still be seen. (Try  $x = 16$  when you have finished the table.)

You should have noticed from this table that the bigger the  $x$ -value, the bigger the gradient. In fact, this is obvious when you know that the gradient of the tangent to  $y = x^2$  is equal to twice the  $x$ -value, i.e.  $\frac{dy}{dx} = 2x$ .

### Rates of Change

The gradient of the tangent,  $dy/dx$ , **also measures a rate of change**. You will have noticed that as you increase the  $x$ -value by dragging the cursor over to the right, the rate at which  $y$  increases for equivalent changes in  $x$  itself increases, i.e. there is an increase in the rate of change of  $y$  with respect to a change in  $x$  as  $x$  increases.

The rate of change of  $y$  with respect to  $x$  **IS**  $\frac{dy}{dx}$  and this idea of rate of change plays an important part when looking at time dependent systems in science and technology (as will be seen in other applets available from *MathinSite*).

So, for example, on  $y = x^2$ , when  $x = 2$  the tangent's gradient (i.e. the rate at which  $y$  changes as  $x$  changes) is 4 and when  $x = 8$  the tangent's gradient (i.e. the rate at which  $y$

changes as  $x$  changes) is 16. You can see that  $y$  changes more rapidly with respect to  $x$  as  $x$  increases. This is obvious when you remember that here  $dy/dx$  is twice whatever is the  $x$  value anyway.

Use the applet to write down:

the rate of change of  $y$  with respect to  $x$  when  $x = 2.3$  is .....

the rate of change of  $y$  with respect to  $x$  when  $x = 7.55$  is .....

the rate of change of  $y$  with respect to  $x$  when  $x = 12.35$  is .....

[You can easily corroborate these answers using  $dy/dx = 2x$ ]

### The Second Differential

You may have noticed from the above table that the values of  $d^2y/dx^2$  remain at a constant value of 2 for the curve  $y = x^2$ , no matter what the value of  $x$  (as seen also from the graph of  $d^2y/dx^2$ ). Knowing about rates of change, the reason for this is easy to understand.

In the notation  $\frac{dy}{dx}$ , each of the separate letters is assigned a specific meaning.  $\frac{dy}{dx}$  reads as:

$d$  = "the rate of change of"

$y$  = "y"

$d$  = "with respect to" (or "w.r.t.")

$x$  = "x"

so $\frac{dy}{dx}$ reads as "the rate of change of $y$ w.r.t. $x$ "
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Now since  $\frac{d^2y}{dx^2}$  can be written as  $\frac{d}{dx}\left(\frac{dy}{dx}\right)$ , then the second derivative can be read as

"the rate of change of  $\frac{dy}{dx}$  with respect to  $x$ ", i.e.  $\frac{d^2y}{dx^2}$  is the gradient of the gradient graph!

Have a look at the gradient graph (the graph for  $\frac{dy}{dx}$ ) and use the applet to write down the gradients of this graph at

$x$	$\frac{d^2y}{dx^2}$
3.20	
6.95	
12.10	

i.e. the gradient of the tangent of the gradient of the tangent graph of  $y = x^2$  (i.e.  $\frac{d^2y}{dx^2}$ ) is

always equal to ..... - the  $\frac{dy}{dx}$  graph has a constant slope.

This tutorial sheet has introduced plenty of theory as well as helping you through the workings of the applet. Take a break before going on to the second tutorial sheet for Differentiation 2: "y = sin x, concavity and stationary points".