Name ………………………………………… Course ……………………………….

Print and use this sheet with ***MathinSite***'s 'Diff of Basic Functions' applet to have a permanent record of your work.

Differentiation 2: Differentiation of some Basic Functions

The Software Tutorial 2:

y = sin x, concavity and stationary points

Learning Outcomes

After using the applet and *this* tutorial sheet you should

- develop, through experimentation, a deeper insight into the differentiation of sine functions
- know the condition that determines whether a curve "opens" upwards or down
- recognise ***stationary points*** and their associated conditions
- be able to answer "what if ...?" questions about the 1st and 2nd differentials of the sine function.

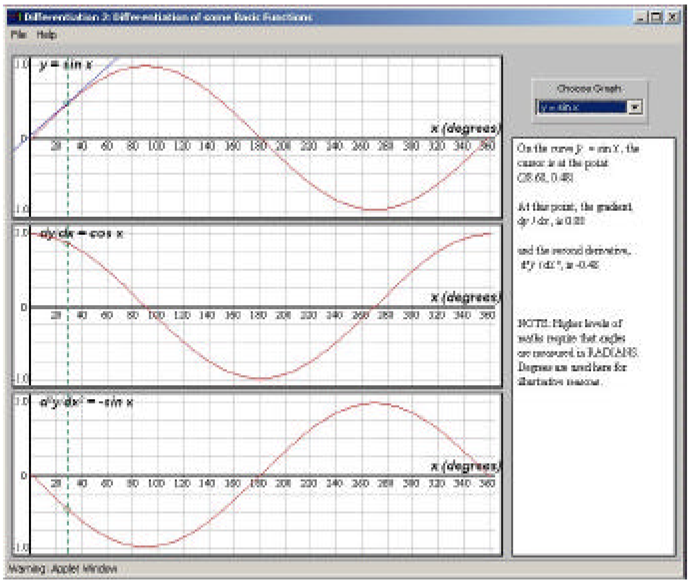
These outcomes are extra to those already listed in the first tutorial sheet.

Introduction

Before using this tutorial sheet it is recommended that you should have used the applet, "Differentiation 1: From Chords to Tangents" - also available from the ***MathinSite*** web site - and the tutorial sheet, "The Software Tutorial 1: $y = x^2$".

Running the applet

You've had instructions already in the first tutorial sheet on how to run the applet so that won't be repeated here. After the applet is loaded, pick "y = sin x" from the drop down menu in the upper right of the applet window. You will see the display similar to that shown here (although the cursor position may be in a different position).



The Tutorial

The applet now shows the graph y = sin x in the top window, again with the tangent to the curve superimposed as a blue line. The second graphics window shows the graph of the derivative, namely dy/dx = cos x and the third shows the second derivative graph, namely d^2y/dx^2 = - sin x (sine again!). Superimposed on all three is the dotted cursor line.

Remember that $\frac{dy}{dx}$ represents the gradient of the tangent (or slope) of the original curve and that $\frac{d^2y}{dx^2}$ represents *the rate of change of the gradient of the tangent*

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The graph $y = \sin x$

[*In passing, it shouldn't have escaped your notice that for* $y = \sin x$, *the differential curve is the cosine curve and the second differential curve takes you back to the original sine curve, albeit MINUS sine.*]

'Grab' the cursor line in the top graphics window and move it slowly left and right. Notice how the blue tangent line moves along with the cursor – and how the *gradient*, or slope, of the blue tangent line changes.

Now use the applet to complete the following table.

Cursor is at $x =$	Cursor is at $y =$	$\frac{dy}{dx} =$	$\frac{d^2y}{dx^2} =$
0^0			
30^o			
45^o			
60^o			
90^o			
120^o			
180^o			

'Grab' the cursor line to move it to the values shown in the table (your computer's processor may not allow you to get all of these values exactly – write in the closest values you can).

Read the values for the *x* and *y*-coordinates and the values of the first and second differentials from the text area on the right of the applet window and put them in the table.

What do all the $\frac{dy}{dx}$ values have in common for $0 \le x < 90$? They're all………

What do the values of $\frac{dy}{dx}$ have in common for $90 < x \le 180$? They're all………

What range of values does $\frac{dy}{dx}$ take in $0 \le x \le 180$? from …… to ……

So in the range $0 \le x \le 180$, is $\frac{dy}{dx}$ increasing or decreasing? …………………

In $0 \le x < 180$, is the gradient of the tangent increasing or decreasing? …………………

Throughout the range $0 < x < 180$, the sign of $\frac{d^2y}{dx^2}$ is …………………

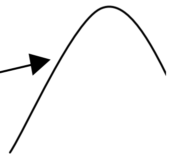
In $0 < x < 180$, the curve opens …(*delete as appropriate*)→ upwards/downwards

From your answers above you should have noted the following important points:

- the slope of the tangent to $y = \sin x$ reduces from +1 at 0^o, through zero at 90^o, down to –1 at 180^o, decreasing all the way. This is reflected in the graph of *dy/dx*, which reduces all the way from +1 to –1 through the range $0 \le x \le 180$.
- In $0 \le x < 90$, the gradient is always positive (and the slope of the curve is 'uphill')
  - ➢ rate of change of *y* w.r.t *x* is positive i.e. $\frac{dy}{dx} > 0$ when the curve is increasing
- In $x = 90$, the gradient is zero (and the curve is horizontal at this point)
  - ➢ i.e. $\frac{dy}{dx} = 0$ at points where the curve is neither increasing nor decreasing

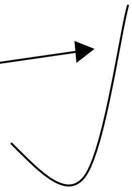
- In $90 < x \le 180$, the gradient is negative (and the slope of the curve is 'downhill')
  - rate of change of y w.r.t x is negative i.e. $\dfrac{dy}{dx} < 0$ when the curve is decreasing
- In $0 < x < 180$, the slope of the *dy/dx* graph is downhill (negative) all the way – the *rate of change of dy/dx* is *negative* - and so $\dfrac{d^2y}{dx^2} < 0$ in this range (even when *dy/dx*= 0). Also note, in $0 < x < 180$ the sine curve is ***concave downwards***.

From this last point you can see that as a curve rises to a peak then falls, its gradient reduces from a positive value through zero to a negative value. Also, note that the curve is concave downwards. This leads to the general point



If $\dfrac{d^2y}{dx^2} < 0$ at any point on a curve, at that point the curve is concave downwards.

Similarly, if a curve descends to a trough then rises, its gradient increases from a negative value through zero to a positive value. Also, note that the curve is concave upwards. This leads to the general point



If $\dfrac{d^2y}{dx^2} > 0$ at any point on a curve, at that point the curve is concave upwards.

The case when d^2y/dx^2 equals zero

Now that you can determine whether a curve opens upwards or downwards, use the applet to complete the following table and comment.

Cursor at x =	Cursor at y =	$\dfrac{dy}{dx} =$	$\dfrac{d^2y}{dx^2} =$
0^o			
180^o			
360^o			

At 180^o, does the curve open upwards or downwards? ………………….
What happens to the blue tangent line as the cursor is moved through 180^o?

………………….……………….

This point doesn't fit either definition for d^2y/dx^2 in the boxes above. Investigation of the curve shows that it neither opens upwards nor downwards – there is no concavity at all. What's happening on the sine curve is that as you pass through 180^o the curve bends from concave one way to concave the other and, at $x = 180^o$, the sine curve is instantaneously straight having no concavity at all.

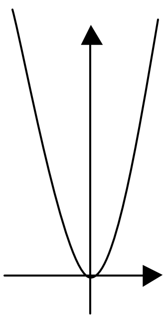
A point at which a curve changes concavity from one direction to another is called a *point of inflection*

At a point of inflection the tangent line crosses the curve and at a point of inflection *the curve is instantaneously a straight line*.

[How can a curve be a straight line "instantaneously", i.e. at only one point? Imagine the sine curve as a road on which you are travelling from left to right. Up to x =180° the steering wheel is turning the car to the right, after x =180° you are turning left. ***AT*** 180° the steering wheel passes through the STRAIGHT-ahead position. Now if you didn't turn the steering wheel away from this 'point of inflection' case, the road wheels would take the car on in a straight line.]

You may be tempted to speculate then that if $d^2y/dx^2 = 0$, you must have a point of inflection - as happened here. This is not necessarily so as the following examples will show.

[The conditions needed to determine *absolutely* whether a point at which $d^2y/dx^2 = 0$ is a point of inflection, which involve higher order differentials, is outside the scope of this tutorial sheet.]



(1) The curve $y = x^4$, looks similar to a parabola , i.e.
and passes through the origin, (0, 0). Complete this table:

$y =$	$\dfrac{dy}{dx} =$	$\dfrac{d^2y}{dx^2} =$	At $x = 0$, $\dfrac{d^2y}{dx^2} =$
x^4			

Hopefully in the above table you found that $d^2y/dx^2 = 0$ at (0, 0). Have another look at the graph of $y = x^4$, it shows that the curve does NOT bend one way then the other at the origin. There's no point of inflection at the origin even though at that point d^2y/dx^2 is zero. What you definitely CAN say though is that at $x = 0$, the curve is 'instantaneously' a straight line.

(2) The straight line $y = mx + c$ has d^2y/dx^2 equal to zero at all points, but nowhere does it have a point of inflection since the straight line does not 'bend' one way, then the other – in fact, it doesn't bend at all!

The conclusion here then is

If $\dfrac{d^2y}{dx^2} = 0$ at any point on a curve, then the curve is instantaneously a straight line at that point *BUT it is not necessarily a point of inflection*.

BUT NOTE

As a curve straightens out, the value of $\dfrac{d^2y}{dx^2}$ does not necessarily tend to zero

[See for example the tangent curve in this applet where, as cursor is moved towards 90°, the curve $y = \tan x$ becomes straighter but the value of the second derivative becomes 'large'. This apparent 'anomaly' will be resolved in the "Radius of Curvature" applet – when it is written!]

Maxima and Minima

Use the applet to complete this table for values on $y = \sin x$.

Cursor at x =	Cursor at y =	$\frac{dy}{dx} =$	$\frac{d^2y}{dx^2} =$
90^o			
270^o			

What is the value of $\frac{dy}{dx}$ when $x = 90^o$ and 270^o? ………………….

What is the gradient of the tangent when $x = 90^o$ and 270^o? ………………….

What is the sign of d^2y/dx^2 at 90^o? ………………….

At $x = 90^o$, the curve is concave … <u>upwards/downwards</u>

What is the sign of d^2y/dx^2 at 270^o? ………………….
At $x = 270^o$, the curve is concave … <u>upwards/downwards</u>

What is significant about the point on the curve when $x = 90^o$? It is a ma..……….

What is significant about the point on the curve when $x = 270^o$? It is a mi……….

From your answers above you should have noted the following important points:

- When $dy/dx = 0$ (i.e. the gradient of the tangent to a curve is horizontal), the curve attains a maximum value, a minimum value - or it may even be a (horizontal) point of inflection.
    - A maximum here may only be a 'local' maximum where a maximum is reached 'locally' but elsewhere the curve takes higher values (see the 'Cubic' applet in this series with values of $a = 1$, $b = 0$, $c = -3$ and $d = 1$, for example), or it may be a 'global' max in which there are no higher points on the curve (See the 'Parabola' applet in this series with values of $a = -1$, $b = 0$ and $c = 3$, for example).
    - A minimum here may only be 'local' minimum where 'locally' a minimum is reached but elsewhere the curve takes lower values, or a 'global' minimum in which there are no lower points on the curve.
    - A point of inflection for which $dy/dx = 0$ and $d^2y/dx^2 = 0$ can be seen using the Cubic applet with $a = 1$, $b = c = d = 0$. The pt of inflection is on the curve at (0, 0).
- For a ***local maximum***, the curve must be concave downwards, i.e. $\boldsymbol{d^2y/dx^2 < 0}$
- For a ***local minimum***, the curve must be concave downwards, i.e. $\boldsymbol{d^2y/dx^2 > 0}$
- Points at which $dy/dx = 0$, i.e. maxima, minima OR (horizontal) points of inflection are called ***stationary points***, or ***turning points***.

Or more concisely:

<u>The Second Derivative Test for maximum and minimum points</u>
For the curve $y = f(x)$, **if, at the point $x = x_0$, $dy/dx = 0$ AND …**

…$d^2y/dx^2 < 0$, then there exists a local maximum at $x = x_0$

…$d^2y/dx^2 > 0$, then there exists a local minimum at $x = x_0$

…$d^2y/dx^2 = 0$, then there might be a local max, local min or pt of inflection at $x = x_0$

[the test is inconclusive in this last case]

So, having $d^2y/dx^2 = 0$ doesn't indicate anything conclusive about the nature of stationary points. In this case an alternative test has to be used which investigates the gradient of the tangents (i.e. the signs of dy/dx) either side of the stationary point at $x = x_0$, namely:

<u>The First Derivative Test for maximum and minimum points</u>
For the curve $y = f(x)$, **if, at the point $x = x_0$, $dy/dx = 0$ AND …**

…$dy/dx < 0$ for $x < x_0$ AND $dy/dx > 0$ for $x > x_0$,
<u>then there exists a local minimum at $x = x_0$</u>
…$dy/dx > 0$ for $x < x_0$ AND $dy/dx < 0$ for $x > x_0$,
<u>then there exists a local maximum at $x = x_0$</u>
…$dy/dx < 0$ for $x < x_0$ AND $dy/dx < 0$ for $x > x_0$,
<u>then there exists a pt of inflection at $x = x_0$</u>
…$dy/dx > 0$ for $x < x_0$ AND $dy/dx > 0$ for $x > x_0$,
<u>then there exists a pt of inflection at $x = x_0$</u>

Use the applet to complete the following table:

Cursor at x =	Cursor at y =	$\frac{dy}{dx} =$	$\frac{d^2y}{dx^2} =$
260°			
270°			
280°			

The applet graph shows that the stationary point at 270° is a …………imum
The Second Derivative Test shows this is correct since d^2y/dx^2 ………
The First Derivative Test shows this is correct since

for $x < 270$ dy/dx…………...
for $x = 270$ dy/dx…………...
for $x > 270$ dy/dx…………...

Exercises

1. The applet showed that the curve $y = \sin x$ has a local maximum at $x = 90^o$ and a local minimum at 270^o.
    a. Sketch the curve $y = \sin x$ in the range $-360^0 \le x \le 720^0$ and show any other local maxima and/or minima.
    b. What equation has to be solved to find all these stationary points?
    c. Use this process to find and classify all the stationary points (i.e. max, min or pt of inflection) of $y = 4\sin 3x$ for $0^0 \le x \le 360^0$

2. Use the second (or first) derivative test on the following to determine the position and nature of any stationary points:
    a. $y = -3x + 5$
    b. $y = -2x^2 - 3x + 5$
    c. $x = 3\cos t$
    d. $y = 2x^3 - 3x^2 - 12x + 6$
    e. $y = x^3 - 9x^2 + 27x + 31$
    f. $y = x^2 e^{-3x}$ (you'll need the Product Rule for differentiation for this one…)
    g. $y = \dfrac{2x^2}{3x-1}$ $(x \ne \frac{1}{3})$ (…and the Quotient Rule for differentiation here)

3. During the completion of this tutorial sheet, it was stated that for the curve $y = x^4$, the value of d^2y/dx^2 at $x = 0$ was zero. What is the value of d^2y/dx^2 at $x = 0$ for $y = x^2$? What do these two results mean in terms of what happens to each curve around $x = 0$?

4. For each of Q. 2 parts a, b, d and e, determine whether the curve opens concave upwards or concave downwards at the point on the curve where $x = 1$.