

Differentiation 2: Differentiation of some Basic Functions

The Software Tutorial 3: $y = \cos x$, $y = \tan x$ and $y = e^x$

Infinites, Asymptotes and Discontinuities



Learning Outcomes

After using the applet and *this* tutorial sheet you should

- develop, through experimentation, a deeper insight into the differentiation of cosine, tangent and exponential functions
- be aware of discontinuities and asymptotes of a graph
- be able to answer “what if ...?” questions about the 1st and 2nd differentials of these functions.

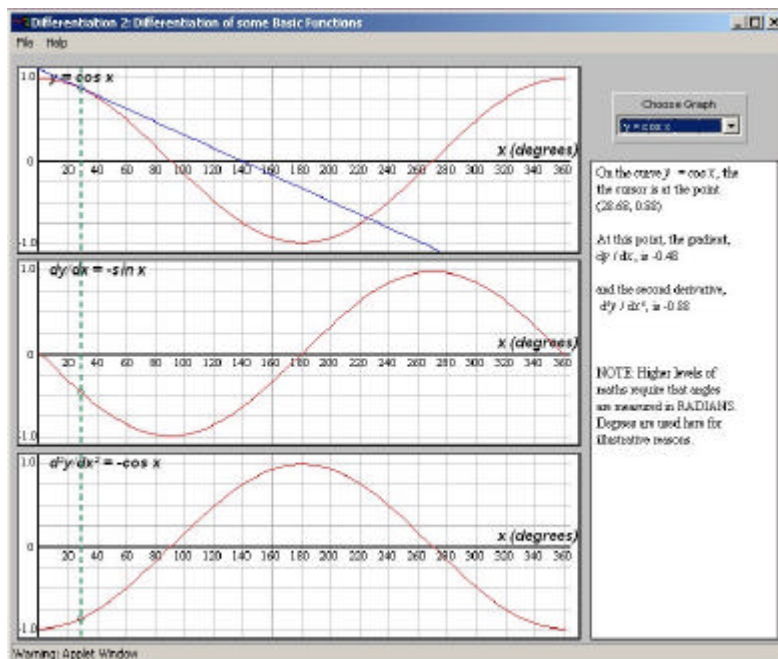
These outcomes are extra to those already listed in the first and second tutorial sheets.

Introduction

Before using this tutorial sheet you should have used the applet, “Differentiation 1: From Chords to Tangents” - also available from the *MathinSite* web site - and the applet and tutorial sheets, Differentiation 2: Differentiation of some Basic Functions, “The Software Tutorial 1: $y = x^2$ ” and “The Software Tutorial 2: $y = \sin x$ ”.

Running the applet

You've already had instructions in the first tutorial sheet on how to run the applet so that won't be repeated here. After the applet loads, pick “ $y = \cos x$ ” from the drop down menu in the upper right of the applet window. You will see a display similar to that shown here (although the cursor position may be different).



The Tutorial

The applet now shows the graph $y = \cos x$ in the top window, again with the tangent to the curve superimposed as a blue line. The second graphics window shows the graph of the derivative, namely $dy/dx = -\sin x$ and the third shows the second derivative graph, namely $d^2y/dx^2 = -\cos x$ (cos again!). Superimposed on all three is the dotted cursor line.

Remember that $\frac{dy}{dx}$ represents the gradient of the tangent (or slope) of the original curve and that $\frac{d^2y}{dx^2}$ represents the rate of change of the gradient of the tangent

The graph $y = \cos x$

[In passing, you should have noticed that for $y = \cos x$, the differential curve is the minus sine curve and the second differential curve takes you back to the original cosine curve, albeit MINUS cosine.]

Use the applet to complete the following table for $y = \cos x$.

Cursor is at $x =$	Cursor is at $y =$	$\frac{dy}{dx} =$	$\frac{d^2y}{dx^2} =$
0°			
30°			
45°			
60°			
90°			
120°			
180°			
270°			

'Grab' the cursor line to move it to the values shown in the table (your computer's processor may not allow you to get all of these values exactly – write in the closest values you can).

Read the values for the x and y -coordinates and the values of the first and second differentials from the text area on the right of the applet window and put them in the table.

What do all the $\frac{dy}{dx}$ values have in common for $0 < x < 180$? They're all.....

What do the values of $\frac{dy}{dx}$ have in common for $180 < x < 360$? They're all.....

Throughout the range $90 < x < 180$, the sign of $\frac{d^2y}{dx^2}$ is

In $90 < x < 180$, the curve is concave...upwards/downwards ←delete as appropriate

Maxima and Minima on $y = \cos x$ in the range 0° to 360° inclusive

Use the applet to complete this table for values on $y = \cos x$.

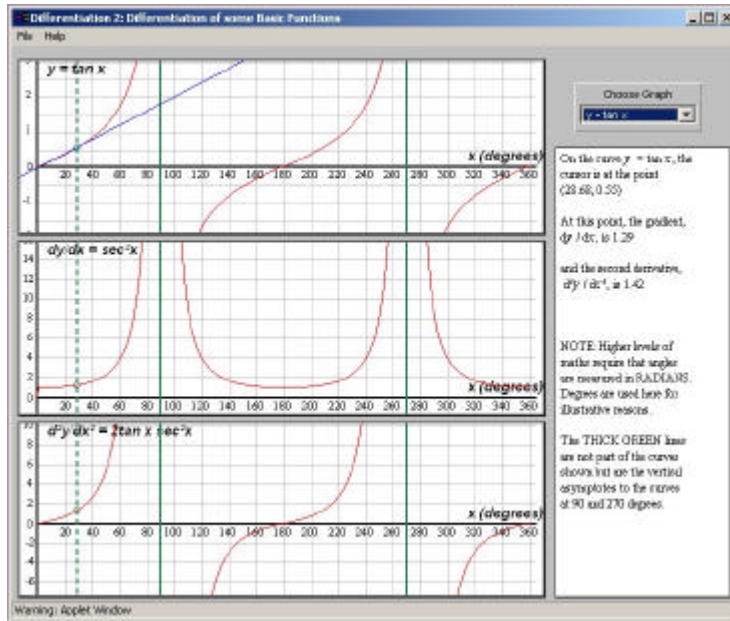
Cursor at $x =$	Cursor at $y =$	$\frac{dy}{dx} =$	$\frac{d^2y}{dx^2} =$
0°			
180°			
360°			

So in the range 0° to 360° inclusive $y = \cos x$ has twoima
and oneimum

Using the applet to deepen insight into maxima, minima, points of inflection and concavity has already been covered in the previous tutorial sheet covering $y = \sin x$.

The graph $y = \tan x$

Choose " $y = \tan x$ " from the drop down menu in the upper right of the applet window. You will see a display similar to that shown here (although the cursor position may be different). Notice that the graphs of $y = \tan x$ and its first and second derivatives, as well as showing the vertical dotted green cursor line and the blue tangent line, also show solid vertical lines at $x = 90^\circ$ and $x = 270^\circ$. These vertical lines are NOT part of the graphs. Their purpose will be explained now.



If not already there, move the vertical dotted cursor line down to $x = 0$ and then slowly move it towards $x = 90^\circ$. As the cursor is moved, the first and third graphs begin to scroll downwards to keep showing the cursor point on each curve. Further towards 90° the second graph begins to scroll too. Passing through $87, 88, 89^\circ$ to 90° , each of the three curves' values (and, indeed, their gradients) become "very large".

Use the applet to complete the following table:

Cursor at $x =$	Cursor at $y =$	$\frac{dy}{dx} =$	$\frac{d^2y}{dx^2} =$
86°			
87°			
88°			
89°			
90°			

'Grab' the cursor line to move it to the values shown in the table (you'll need a steady hand!). Your computer's processor may not allow you to get all of these values exactly – write in the closest values you can.

Read the values for the x and y -coordinates and the values of the first and second differentials from the text area on the right of the applet window and put them in the table.

Infinity - Mathematicians versus Engineers

"Infinity"? What is "Infinity"? You can see that as the cursor passes through $87, 88, 89^\circ$, the y value becomes increasingly larger – so much so that AT 90° , the value is sooo large that the mathematicians say that it is 'undefined', whereas the engineers are more likely to say that the value approaches (or even, is) 'infinity'. Unfortunately, even the star ship USS Enterprise flying at "Warp Factor 9" cannot possibly reach infinity – you may think you are AT infinity, just as someone flies past you! This leaves mathematicians rather reluctant to use a term for something that has not ever been seen (or defined). For engineers, however, infinity just means a "very large" number. Infinity is usually given the symbol, ∞ .

Asymptotes

As the cursor moves towards 90° , each curve approaches but (theoretically) never touches an invisible vertical line through $x = 90^\circ$. In the applet here, though, these vertical lines have been drawn to illustrate (for illustration only) and are called **asymptotes**. Obviously our three curves here merge in to these asymptotes due to the thickness of the lines – theoretically, asymptotes and curves are infinitesimally thin and no matter what the level of magnification you would not see a curve touch its own asymptote. If you have used the “Exponential Function” applet in this series, you will have seen asymptotes before. The graph of $y = e^{-x}$ has the x -axis as a horizontal asymptote. The curve $y = e^{-x}$ approaches (but never touches) the x -axis as x becomes large. You may wish to mark this here and come back to it next time you load up the “Exponential Function” applet.

On the graph of $y = \tan x$, in the range $0 \leq x \leq 360$,
there is a second asymptote at $x =$

Discontinuities

Run the cursor from left to right from 80° to 90° . As the graph approaches 90° from the left (or, as $x \rightarrow 90^\circ$), the y -value (i.e. $\tan x$) increases off to PLUS infinity. Now move the cursor to about 100° and move it slowly from right to left down to 90° . Now as $x \rightarrow 90^+$ (i.e. approaches 90 from the positive side), $\tan x$ decreases down to MINUS infinity. Passing through 90° from left to right, the value of $y = \tan x$ changes from $+\infty$ to $-\infty$, *instantaneously*. If you were able to draw this, you would keep your pen on the page as you draw up to $+\infty$, *then you have to take your pen off the page*, place it down to $-\infty$ and continue drawing. In cases like this where you have to take your pen off the page to continue drawing at a different height, you say that the curve has a **discontinuity** (at 90° , 270° , etc. for $y = \tan x$).

On the graph of $y = \tan x$, in the range $0 \leq x \leq 360$,
there is a second discontinuity at $x =$

Is $y = \tan x$ a straight line as it approaches its asymptote?

Look at the applet window. Move the cursor close to 90° or 270° . What happens? It straightens out. It has to. It is approaching a straight line (the asymptote itself) so it has to become straight.

Use the applet to complete this table:

Cursor at $x =$	$\frac{d^2 y}{dx^2} =$
0°	
87°	
88°	
89°	
180°	

Knowing what you have seen in this and the previous tutorial sheet about what happens when $d^2y/dx^2 = 0$, what can you say about the curve of $y = \tan x$ at 0° and 180° ?

.....
Look at the curve of $y = \tan x$, is it becoming straighter as you approach 90° ? Yes / No

Does d^2y/dx^2 become zero as $x \rightarrow 90^\circ$? Yes/No

Print and use this sheet with *MathinSite's* 'Diff of Basic Functions' applet to have a permanent record of your work.

This last task highlights very effectively a point raised in the previous tutorial sheet:

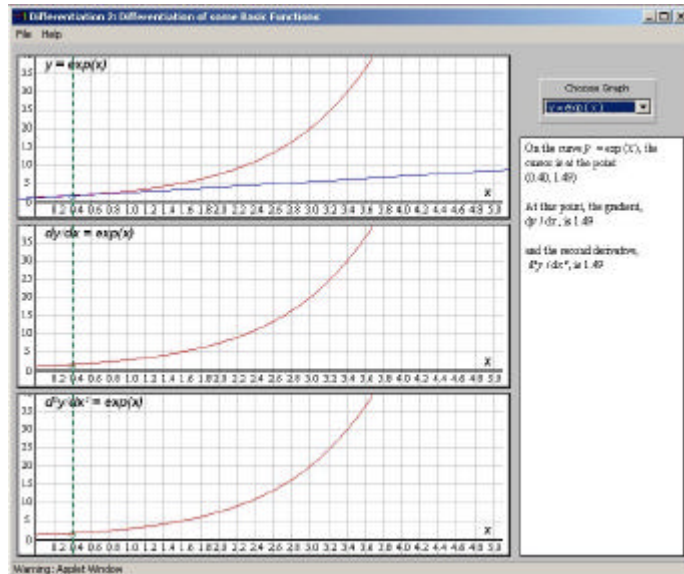
If $d^2y/dx^2 = 0$ at any point on a curve, the curve is instantaneously straight at that point
BUT

The fact that a curve may be straight does not necessarily imply that d^2y/dx^2 will be zero

Why this is so is resolved in the "Radius of Curvature" applet.

The graph $y = e^x$ (or $y = \exp x$)

Choose " $y = \exp(x)$ " from the drop down menu in the upper right of the applet window. You will see a display similar to that shown here (although the cursor position may be different). Notice that the graphs of $y = \exp(x)$ and its first and second derivatives *are all the same!* The exponential function $y = e^x$ is the only function whose differential equals the original function.



[This results in the famous 'differential equation' $\frac{dy}{dx} = y$, which appears in many science and engineering applications.]

From the applet graphics windows, note the following points about this particular exponential curve:

- Following the curve to the left of $x = 0$, the curve slowly descends to, but never attains, the value zero, i.e. the x -axis is an asymptote to $y = e^x$.
- Since the curve for dy/dx is the also of the form e^x , dy/dx never equals zero, so the graph of $y = e^x$ never reaches a local maximum or a minimum.
- On this curve, d^2y/dx^2 **does** tend towards zero as $x \rightarrow -\infty$, therefore you can say with certainty that the curve straightens out as $x \rightarrow -\infty$. (It may or may not straighten out as $x \rightarrow +\infty$, but you can't tell this from the value of d^2y/dx^2 which itself becomes 'large' as x becomes large.)
- $d^2y/dx^2 = e^x$ is always positive, so the curve $y = e^x$ is always concave upwards.

Exercises

1. The applet showed that the curve $y = \cos x$ has a local maximum at $x = 0^\circ$ and a local minimum at 180° .
 - a. Sketch the curve $y = \cos x$ in the range $-360^\circ \leq x \leq 720^\circ$ and show any other local maxima and/or minima in this range.
 - b. What equation has to be solved to find all these stationary points?
 - c. Use this process to find and classify all the stationary points (i.e. max, min or horizontal points of inflection) of $y = 5\cos 2x$ for $0^\circ \leq x \leq 360^\circ$

2. This tutorial sheet didn't show whether the curve $y = \tan x$ has any local maxima or local minima.
 - a. How can you tell from the second graph on the applet window that $y = \tan x$ has no stationary values?
 - b. With the help of the third graph determine the position of any points of inflection.
 - c. Sketch the curve $y = \tan x$ in the range $-360^\circ \leq x \leq 720^\circ$. Notice the 'periodicity' of the tangent function.

3. Sketch the three graphs that would have resulted in the applet's three graphics windows if it had used $y = e^{-x}$ instead of $y = e^x$. Rewrite the four 'bullet points' at the bottom of page 5 for this case.