The Exponential Function, $y = A + Be^{kt}$. The Software Tutorial

Learning Aims

- To develop, through experimentation, an understanding of exponential functions and a feel for the effect of changing their parameters.
- To enable questions about the exponential function of a "what if ...?" nature to be answered.

Introduction

The exponential function has important applications in the mathematical analysis of topics in, amongst others, science, technology and business studies. For example, population dynamics, vibrations of mass/spring/damper systems and compound interest highlight areas where exponential growth or exponential decay occurs. Since it is frequently used to model time dependent systems, the independent variable used here is *t* (for time) rather than *x*.

This package allows the user to investigate a generalised form of the exponential function,

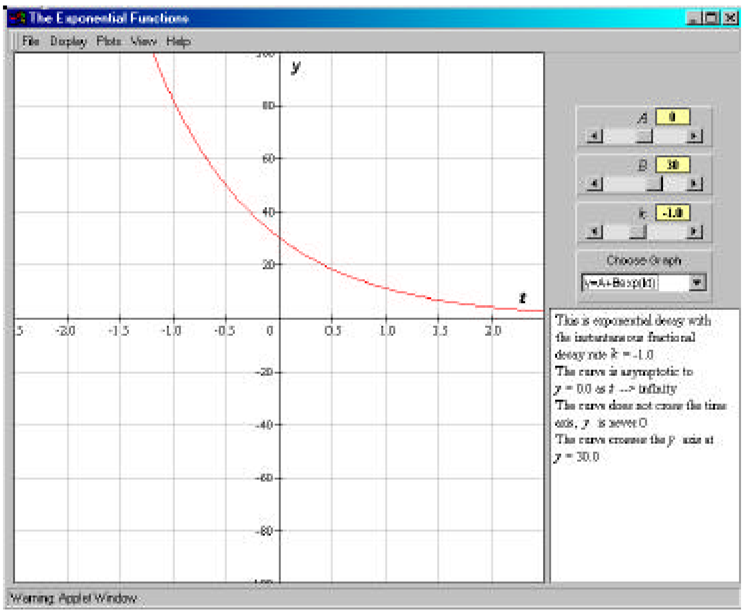
$$y = A + Be^{kt}$$

or, as is sometimes written, $y = A + B\exp(kt)$

This latter format is used in the software, and occasionally in written texts, for typographical reasons. The more commonly used format is $y = A + Be^{kt}$. In this tutorial sheet you will be exposed to both formats to ensure familiarity with the use of e^{kt} and $\exp(kt)$.

Loading the Software

The applet, "Exponential Functions" is obtained from the web site from which this work sheet was downloaded. When the applet is loaded, you will see something similar to the display shown here.



Running the applet

The default function $y = A + B\exp(kt)$ will be shown in the graphics window with the default parameter values $A = 0$, $B = 30$ and $k = -1.0$.

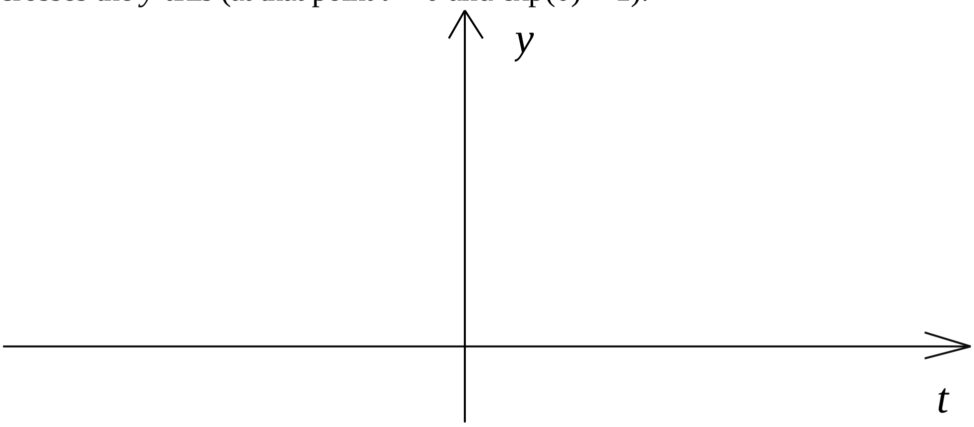
The three slider bars on the right of the applet window allow the user to change each of the above parameters. Clicking the arrow boxes at either end of the slider bar effect *small changes* in the parameter values. Clicking in the main part of the box of the slider bar makes *larger changes. Clicking, holding and moving the <u>slider</u> in the slider bar can make* ***any*** *sort of change.*

Changing the values by moving the sliders is the only means of user-input.

The software gives you the option of showing single plots or multiple plots (under 'Plots' in the pull down menu structure at the top of the applet window). It also allows you to 'drag' the plot around (to see areas not covered by the original window on the plot) by holding down the mouse button whilst at the same time moving the mouse over the plot. The plot can be recentred at any time using the 'Centre on Origin' option, under 'Display'. You can reset the software at any time to the default values (those used when the program loads – given above) using 'Reset' from the 'Plots' menu.

<u>The Tutorial</u>

<u>**Changing *k***</u> The display shows the graph of $y = A + Be^{kt}$ with default values $A = 0$, $B = 30$ and $k = -1$. From the "Plot" menu at the top of the screen choose "Multiple Plots" and superimpose the four curves for $A = 0$, $B = 30$ and $k = -2, -3, -4$ and -5 by clicking four times to the *left* of the slider in the k slider bar. Sketch the five curves on the axes below, labelling each and indicating on the graph the value the y-intercept, i.e. where the curve crosses the y-axis (at that point $t = 0$ and $\exp(0) = 1$).



Write down the equations in full for the five curves you have just sketched.

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Describe the overall shape of the curve(s) and, in particular, the effect of changing the value of k from $k = -1$ to -5. You should include a comment on how the gradients of the curves are affected by changing k and hence the effect that k has on the *growth rate.* (The official name for k is "the instantaneous fractional growth rate".)

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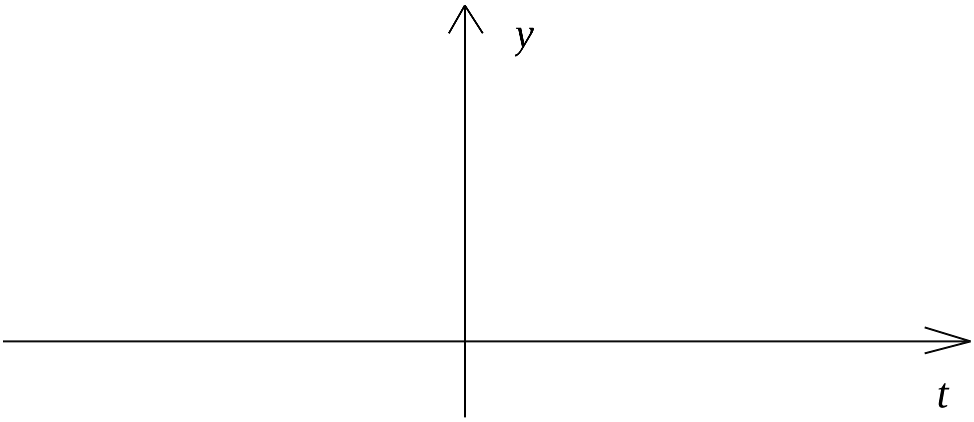
The above curves are all examples of *exponential decay*.

Now determine the effect of making k positive. From the "Plots" menu choose "Reset" and move the k slider to 1 and press 'Clear Multiple Plots' from the "Plots" menu. Superimpose with the cases when k = 2, 3, 4 and 5 by clicking in the k slider bar to the *right* of the slider. Sketch and label the curves below:



Describe the overall shape of the curve(s) and, in particular, the effect of changing the value of k from k = 1 to 5. You should include a comment on how the gradients of the curves are affected by changing k and hence the effect that k has on the *growth rate.*

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The above curves are all examples of *exponential growth.*

Describe what you notice about the curves y = 30exp(-t) and y = 30exp(t), or, similarly, y = 30exp(-2t) and y = 30exp(2t), etc.

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Did you notice that we didn't try k = 0? Before you do, guess what the curve will look like. Bear in mind that k represents a growth/decay rate, what happens if k = 0? Plot it and see. Would you describe the curve as growth or decay? Why?

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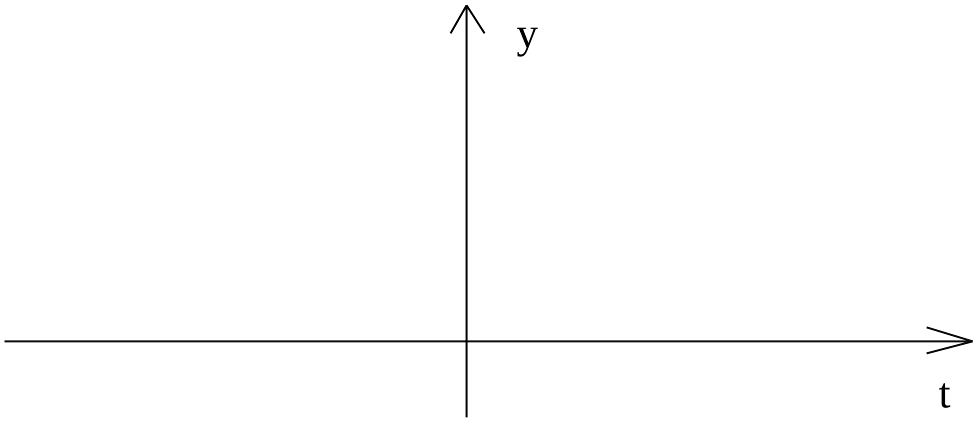
Changing *B* Now use the slider bars to set up A = 0, B = 10 and k = 1.5, clear the screen and plot this curve. Now superimpose the curves for which B = 20, 30, 40 … as far as you need to sketch the family of curves obtained and to answer the following question. Indicate on your diagram the value(s) of the y-intercept(s) and label each curve. Note that the curves are all examples of exponential growth since k is positive.



Describe the effect of increasing the value of *B* from *B* = 10.

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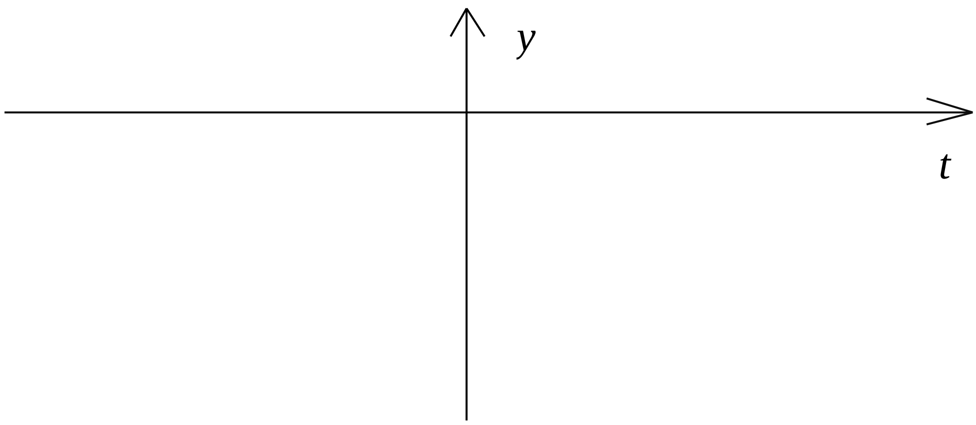
Now consider *B* with negative values. Try *B* = -10, -20, -30, -40, ... Sketch and label the curves along with their *y*-intercepts.



Describe what you notice about the curves *y* = 10exp(1.5*t*) and *y* = -10exp(-1.5*t*), or, similarly, *y* = 20exp(-1.5*t*) and *y* = -20exp(1.5*t*), etc.

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Did you notice that we didn't try *B* = 0? Before you do, guess what the curve will look like. Bear in mind that that this time *k* is not zero, what happens if *B* = 0? Plot it and see. Would you describe the curve as growth or decay? Why?

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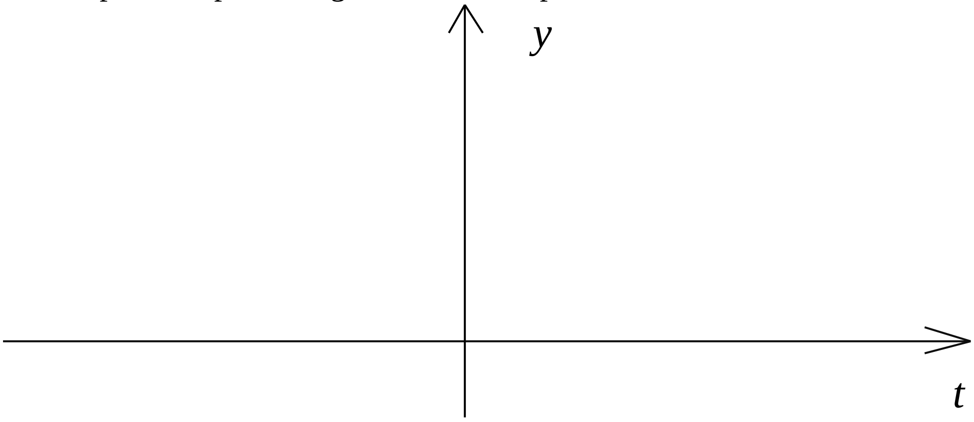
<u>Changing *A*</u> Now use the slider bars to set *A* = 0, *B* = 10 and *k* = 1.5 and clear the screen to see this curve. Superimpose the curves for which *A* = 10, 20, 30, 40 … as far as you need to sketch the family of curves obtained and to answer the following question. Indicate on your diagram the values of the y-intercepts and label each curve. Note that the curves are all examples of exponential growth since k is positive.



Describe the effect of *increasing* the value of A.

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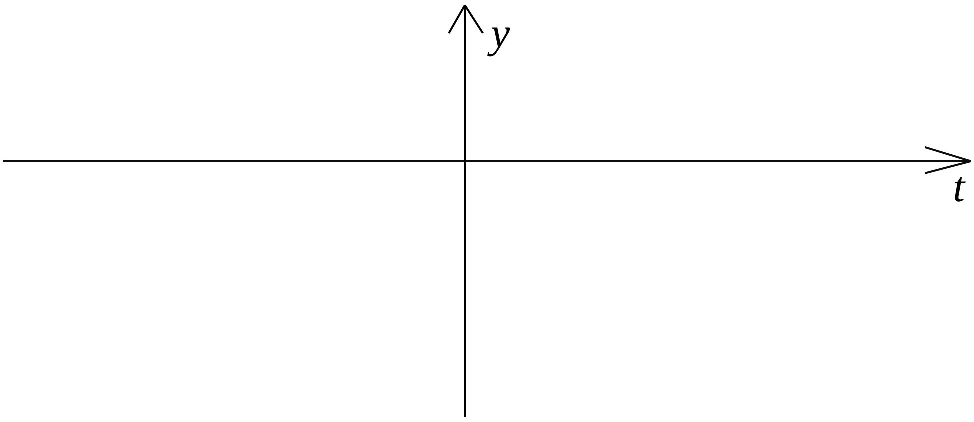
Now consider A with negative values. Try A = -10, -20, -30, -40, ... Sketch and label the curves along with their y-intercept(s).



Describe the effect of reducing the value of A (i.e. making it more negative).

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<u>The Horizontal Asymptote</u> Notice how, whether the curve represents exponential growth ($k > 0$) or decay ($k < 0$), the curve approaches, but (theoretically) never reaches, the horizontal line given by $y = A$, this line is called an *asymptote* and the curve is said to be *asymptotic* to this line.

For the following curves, give the equations of the asymptotes:

$y = A + B\exp(kt)$ y = -10 - 20exp(-3t) y = 50 - 30exp(2.5t) $y = -15 + 40e^{2t}$

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The y-intercept for $y = A + Be^{kt}$ is the value of y for which $t = 0$. Write down the value of the y-intercept for the four equations above.

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Describe the effect on the graph of $y = A + Be^{kt}$ of changing the *sign* of k.

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Describe the effect on the graph of $y = A + Be^{kt}$ of changing the *sign* of B.

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Describe the effect on the graph of $y = A + Be^{kt}$ of changing A.

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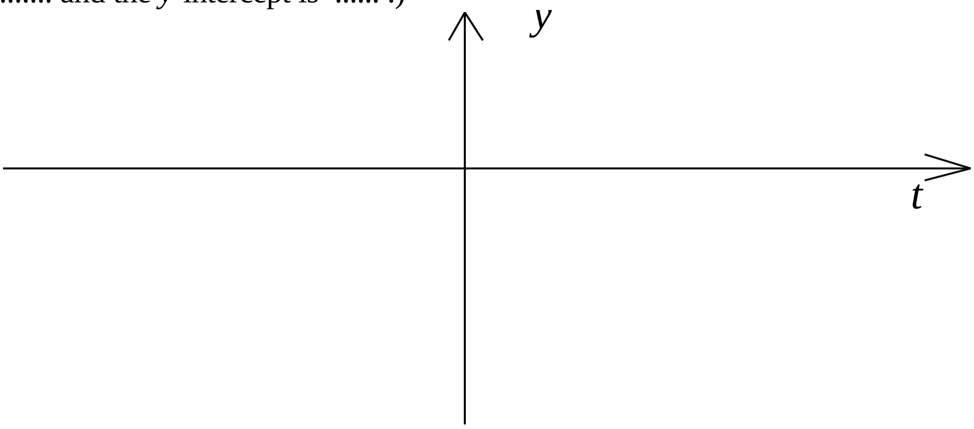
Try this *without* using the software. Sketch the graph of $y = 10 - 20\exp(-3t)$.
(Hint: $\exp(-3t)$ represents exponential .... , the -20 has the effect of .... , the +10 has the effect of .... , the horizontal asymptote is $y =$ .... and the y-intercept is ....)



Use the software to check your answer.

When *sketching* such curves, it is best to think in 'reverse order'. First, does the k indicate growth or decay? Second, does the B indicate a reflection in the t-axis or not? Third, does the A push the curve up or pull it down? You should also determine the y-intercept (by putting $t = 0$) and noting the position of the horizontal asymptote ($y = A$).

So, do the same for the following:

$y = 25\exp(3t)$	$y = 20 + 25e^{3t}$	$y = 25\exp(-3t)$
$y = -10e^{2t}$	$y = -20 - 55e^{-5t}$	$y = 10 - 40\exp(-2t)$
$y = -15 + 25\exp(4t)$	$y = 20 + 15e^{0t}$	

You can determine where $y = A + Be^{kt}$ crosses the t-axis (if at all) by rearranging this equation to make t the subject. Show that this results in

$$t = \frac{1}{k}\ln\left(\frac{y - A}{B}\right).$$

For what values of the parameters A, B and/or k will this equation 'not work'? Why? What, graphically, do these cases refer to? Use the software to corroborate your findings, including evaluating the co-ordinates of some points where the curves do cross the t-axis.