The LRC Series Circuit

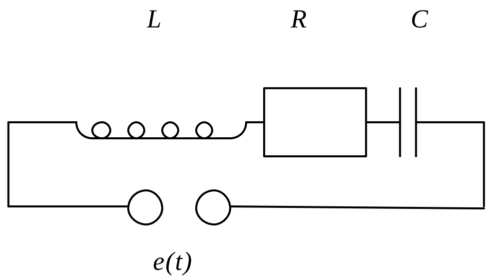
Theory Sheet 2
The Three Types of Complementary Function

Learning Outcomes, Prerequisites & Background

These are all outlined in Theory Sheet 1

> The applet accompanying these work sheets can be found on the ***MathinSite*** web site at http://mathinsite.bmth.ac.uk/html/applets.html.

The LRC series circuit



The governing differential equation for this circuit in terms of current, i, is

$$L\frac{d^2i}{dt^2}+R\frac{di}{dt}+\frac{1}{C}i=\frac{de(t)}{dt}$$

Finding the Complementary Function (CF) of the Differential Equation

Investigation of the CF alone is possible whether using the Assumed Solution method or the Laplace Transform method (both of which were outlined in Theory Sheet 1).

> **A more comprehensive explanation of these methods can be found in a variety of textbooks.**
> **One such example is "Engineering Mathematics" by Stroud, K.A. and Booth, D.J.**
> **(Fifth edition, published by Palgrave, UK, 2001) See pages 1071 – 1117.**

Method 1 The Assumed Solution method

The general differential equation above, with an assumed solution of $i = Ae^{mt}$, gives rise to the ***auxiliary equation***

$$Lm^2+Rm+\frac{1}{C}=0$$

which is a quadratic equation in m.

Any quadratic of the form $am^2+bm+c=0$ has

(a) 2 real, distinct roots, m_1 and m_2,   (m_1, m_2 negative)
(b) 2 real, equal roots, m (and m)   (m negative)
(c) 2 complex roots, $m_1 = a + bj$, $m_2 = a - bj$ (a negative)

depending on the sign of the *discriminant* $b^2 - 4ac$.

m_1, m_2, m and a will always be negative since the coefficients of the auxiliary equation (involving L, R and C) are always positive.

For the auxiliary equation, then, this gives rise to the following types of CFs:

	Discriminant	***Roots***	***CF***
(a)	$R^2 - \frac{4L}{C} > 0$	m_1 and m_2	$i = Ae^{m_1 t} + Be^{m_2 t}$
(b)	$R^2 - \frac{4L}{C} = 0$	m (and m)	$i = (At + B)e^{mt}$
(c)	$R^2 - \frac{4L}{C} < 0$	$m_1 = a + bj,\ m_2 = a - bj$	$i = e^{at}\left(A\cos bt + B\sin bt\right)$

Method 2 The Laplace Transform method

During the Laplace Transform method of solution, the following compound fraction in s will occur:

$$\bar{i} = \frac{\dots}{\left(Ls^2 + Rs + \frac{1}{C}\right)(\dots)}$$

where the "…" represent expressions in s.

This time, $Ls^2 + Rs + \frac{1}{C} = 0$ is called the ***characteristic equation*** (rather than the *auxiliary equation* in m obtained using the Assumed Solution method).

The characteristic expression, $Lm^2 + Rm + \frac{1}{C}$ will either factorise into

(a) 2 real, distinct linear factors (if the discriminant, $R^2 - \frac{4L}{C} > 0$),

(b) 2 real, equal linear factors (if $R^2 - \frac{4L}{C} = 0$), or

(c) will require the "completing the square" process (if $R^2 - \frac{4L}{C} < 0$).

In which case the following types of partial fractions and their inverse Laplace Transforms will occur (again with m_1, m_2, m and a negative):

	Discriminant	***Type of Partial Fractions***	***Inverse Laplace (CF only)***
(a)	$R^2 - \frac{4L}{C} > 0$	$\bar{i} = \frac{A}{s - m_1} + \frac{B}{s - m_2} + \dots$	$i = Ae^{m_1 t} + Be^{m_2 t} + \dots$
(b)	$R^2 - \frac{4L}{C} = 0$	$\bar{i} = \frac{A}{(s - m)^2} + \frac{B}{s - m} + \dots$	$i = (At + B)e^{mt} + \dots$
(c)	$R^2 - \frac{4L}{C} < 0$	$\bar{i} = \frac{A(s - a) + Bb}{(s - a)^2 + b^2} + \dots$	$i = e^{at}\left(A\cos bt + B\sin bt\right) + \dots$

<u>Interpretation of the CF</u>

Note that the CF is dependent only on the system itself (i.e. it is found using values of L, R and C only and not on the applied voltage).

Since m_1, m_2, m and a are always negative, each of three types of CF involves exponential *decay* and so the CF is a ***transient*** part of the overall solution, that is, it is short-lived. Short-lived is a relative term, so how short-lived is short-lived? This can be found from the ***time constant***, t , for each exponential.

> **Note on time constants**: Any exponential ***decay*** term of the form e^{-kt} ($k > 0$) [or e^{mt} ($m < 0$)] has a time constant defined by $t = 1/k$ [or $t = 1/m$]. t is a useful quantity since $5\times t$ gives a measure of approximately how long the exponential term takes to decay away to zero. In this theory sheet and in ***MathinSite***'s LRC Series Circuit applet, $5\times t$ is used, although some other authors use $6\times t$.
>
> Further information on time constants can be found in the theory and tutorial sheets of ***MathinSite***'s 'Exponential Function' applet.

So the current settles down to a ***steady state*** after a time $5\times t$ approximately.

In the case of $i = Ae^{m_1 t} + Be^{m_2 t}$, the time for the transient to decay is the longer of $5\times\frac{1}{m_1}$ and $5\times\frac{1}{m_2}$.

> The solution $i = Ae^{m_1 t} + Be^{m_2 t} + \ldots$ occurs when $R^2 > \dfrac{4L}{C}$ and can be achieved using a combination of large resistance, small inductance and large capacitance. The transient exponential decay is likely to be 'slow'. This type of response, with large resistance, is called '***heavy damping***'.

> The solution $i = e^{at}\left(A\cos bt + B\sin bt\right) + \ldots$ occurs when $R^2 < \dfrac{4L}{C}$ and can be achieved using a combination of small resistance, large inductance and small capacitance. The transient will exhibit exponential decay with oscillations (courtesy of the sin/cos terms). With comparatively small resistance, this type of response is called '***light damping***'.

> The solution $i = (At + B)e^{mt} + \ldots$ occurs when $R^2 = \dfrac{4L}{C}$. The transient exponential decay here will be the fastest it possibly can be without resulting in oscillations (there are no sin/cos terms). This type of response, the critical boundary between heavy damping and light damping, is called '***critical damping***'.