Name ……………………………………… Course ………………………………

Print and use this sheet in conjunction with ***MathinSite***'s 'LR Series Circuit' applet and tutorial sheets.

The LR Series Circuit

Theory Sheet 1

Learning Outcomes

After using the ***MathinSite*** LR Series Circuit applet and its accompanying tutorial and theory sheets you should

- be aware of the composition of an LR Series circuit
- be able to model the LR Series Circuit mathematically
- be aware of solution methods for the LR circuit's associated differential equation
- be aware of the different types of solution of the LR circuit differential equation
- have developed, through experimentation with the applet, an understanding of how the current in an LR series circuit responds to changes in system parameters and applied voltages
- be able to answer "what if ...?" questions about the LR Series Circuit.

Prerequisites

Before using the applet, this theory sheet and the accompanying tutorial sheets, familiarity with the following mathematics would be useful (but not vital).

- The Straight Line
- The Exponential Function – including the notion of 'time constants'
- Trigonometrical Functions (in particular sines and cosines)
- Differentiation and Integration, and
- The solution of first-order Differential Equations

However, ***even without this knowledge***, just understanding how the circuit responds can help in your appreciation of the mathematics involved. Applets covering most of the above mathematical topics can also be found on the ***MathinSite*** web site at (http://mathinsite.bmth.ac.uk/html/applets.html).

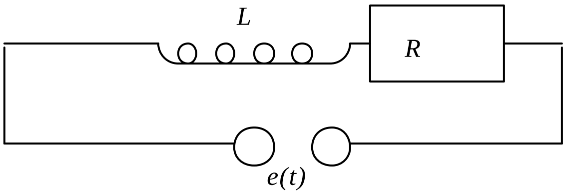
Background

Inductance, L, resistance, R, and capacitance, C, are the building blocks of basic electrical systems studied in first-year undergraduate engineering. A voltage applied to such a circuit (the input) containing these elements will result in a current flow (the output, or response). But what form does the resulting current take?

This applet allows investigation of a circuit containing an inductance and a resistance in series (no capacitor present here – for that case, see the LRC Series Circuit applet). The parameters L and R can be varied, along with the initial current in the circuit and applied voltages (constant, ramp, sinusoidal and exponentially varying voltages can be input). The applet displays the circuit's response as graphical output together with the governing differential equation and the mathematical equation of the resulting current.

Using the LR Series Circuit applet will help give you a feel for how changes of the circuit parameters affect circuit response – both graphically and mathematically.

The LR series circuit



The electrical circuit above shows an inductance, L, and a resistance, R, in series. A current $i(t)$ will flow in the circuit when a voltage $e(t)$ is applied.

Using $i(t)$ and $e(t)$ indicates that current and voltage vary with time (i.e. they are time-dependent variables). Of course this doesn't preclude the possibility that the voltage, for example, could be constant (e.g. the voltage source is a battery).

The drop in voltage (i.e. potential drop) across the resistance is iR (Ohm's Law), and the potential drop across L is $L\frac{di}{dt}$.

Kirchhoff's 2nd Law says that the sum (i.e. addition) of potential drops across all of the non-supply elements in the circuit equals the applied voltage of the supply. So,

$$\boxed{L\frac{di}{dt} + Ri = e(t)}$$

and this a *first-order differential equation*.

Usually such an equation has an associated *initial condition* specifying a value of the current in the circuit when the voltage source is applied (often at time zero). Mathematically, this is sometimes written, $i = i_o$ when $t = 0$, or as $i(0) = i_o$.

Digression on Variables and Parameters In the above circuit $e(t)$ is the *input* to the circuit and it *varies* with time, i.e. it is time dependent. $i(t)$ is the *output* of the circuit once the input voltage has been applied and it, too, is a time dependent variable.

$i(t)$ and $e(t)$ are called ***dependent variables*** since they are dependent here on time, t, which is called the ***independent variable***. Note that in any such circuit, current and voltage will *vary* (unless they are constant) all the time. However, in a particular circuit L and R will always take just the one value for all time. This doesn't mean it is not possible to change either of them - but if they are changed, this results in a completely different circuit and hence a completely different differential equation to solve. Here, L and R are ***parameters*** of the system.

Solving Differential Equations

As with many real-world applications, the LR series circuit is a time-dependent system. Generally, the ***solution*** of such a system's differential equation is the ***output*** of the system and is, in this case, the current, i, at time, t. It is therefore necessary to **solve the differential equation to obtain "*i* = some function in terms of *t*".**

The LR Series Circuit's differential equation can be solved in a variety of ways depending on how complicated is the expression for the applied voltage.

Method 1. When the applied voltage is zero or a constant value, the differential equation is a ***separable equation*** (i.e. it is possible to separate the variables on to each side of the equation)

The Variable Separable Method uses integration to solve **first-order** diff equations:

In any equation of the form $\frac{dy}{dx} = f(x)g(y)$, where the fns of x and y can be separated

rearrange to give $\frac{1}{g(y)}\frac{dy}{dx} = f(x)$ and then integrate both sides with respect to x

so giving $\int \frac{1}{g(y)}\frac{dy}{dx}dx = \int f(x)dx$

This reduces to $\int \frac{1}{g(y)}dy = \int f(x)dx$

So to see how the current builds up in the LR circuit when an applied voltage is switched on, it is necessary to solve the circuit equation

$$L\frac{di}{dt} + Ri = e(t)$$

Suppose that for a particular circuit, L = 1 H and R = 5 Ω and that the governing

differential equation is $\frac{di}{dt} + 5i = 1$ ← i.e. constant applied voltage of 1V.

with $i = 0$ when $t = 0$ ← the initial condition

Rewrite as $\frac{di}{dt} = 1 - 5i$ and separate the variables

$$\int \frac{1}{1-5i}di = \int dt$$

$$-\frac{1}{5}\int \frac{-5}{1-5i}di = \int 1\,dt$$ (make numerator the differential of the denominator)

resulting in $t = -\frac{1}{5}\ln(1-5i) + c$ where c is the constant of integration.

Now use the *initial condition*, namely $i = 0$ when $t = 0$,

so $0 = -\frac{1}{5}\ln(1) + c \quad \Rightarrow c = 0$

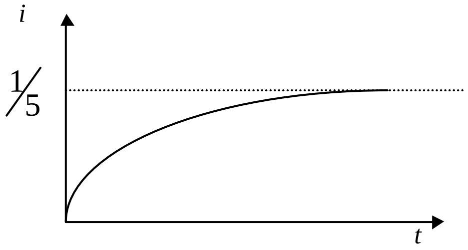
so $t = -\frac{1}{5}\ln(1-5i)$

or, $\ln(1-5i) = -5t$ exponentiate both sides

$1 - 5i = e^{-5t}$ so, finally

$$\underline{i = \frac{1}{5}\left(1 - 5e^{-5t}\right)}$$

This has the solution curve (easily found using the LR Series Circuit applet),



Note that the current settles down to a ***steady-state*** value of one-fifth of an amp and has a ***time constant***, t , of one-fifth of a second - the ***transient*** (that part of the solution relating to the exponential decay) dies away in about one second.

Note on time constants: Any exponential ***decay*** term of the form e^{-kt} $(k > 0)$ has a time constant defined by $t = 1/k$. t is a useful quantity since $5 \times t$ gives a measure of approximately how long the exponential term takes to decay away to zero. In this theory sheet and in ***MathinSite***'s LR Series Circuit applet, $5 \times t$ is used, although some other authors use $6 \times t$.

Further information on time constants can be found in the theory and tutorial sheets of ***MathinSite***'s 'Exponential Function' applet.

When the applied voltage is sinusoidal, exponential or general linear, separating the variables as in the example above becomes impossible. In these cases it is still possible to solve the differential equation by integration using the "Integrating Factor" technique. This method is not covered here.

The LR Series Circuit differential equation is one in which integration techniques of solution can be avoided totally, fortunately, by using the **Laplace Transform** method of solution. If you are studying on an engineering course, it is highly likely that you are, or soon will be, familiar with this method. As an introduction (or reminder) of the method, let's solve the same equation, but this time using Laplace Transforms (LTs).

Method 2. Laplace Transform Solution

$$\frac{di}{dt} + 5i = 1$$

with $i = 0$ when $t = 0$

Same equation and initial condition as before, but this time the Laplace Transform of the equation will be used - found from tables of LTs from formula, or text, books.

REMINDER: *from the definition of the Laplace Transform (LT) and tables of transforms,*

the LT of i(t) is $\bar{i}$,

the LT of $\dfrac{di}{dt}$ *is* $s\bar{i} - i(0)$, *where i(0) is the value of i when t = 0, and*

the LT of 1 is $\dfrac{1}{s}$.

Take the Laplace transform of the differential equation to give

$$s\bar{i} - 0 + 5\bar{i} = \frac{1}{s}, \quad \text{or}$$

$$(s+5)\bar{i} = \frac{1}{s}, \quad \text{so}$$

$$\bar{i} = \frac{1}{s(s+5)} = \frac{1}{5s} - \frac{1}{5(s+5)}$$

by "Partial Fractions".

Now take inverse Laplace Transform to find $i(t)$.

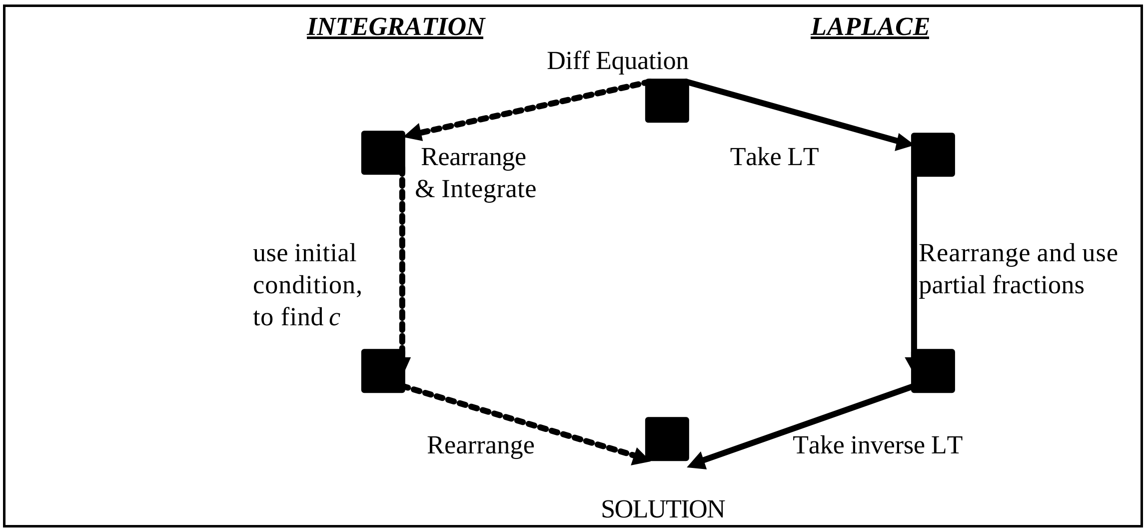
(*These can be found from LT tables reading from the **right**-hand column across to the **left**-hand column.*)

NO CALCULUS!

$$i = \frac{1}{5} - \frac{1}{5}e^{-5t} = \frac{1}{5}\left(1 - e^{-5t}\right) \quad \text{as before.}$$

If you have not seen this method before, don't worry! It's here just to show you that there is another way of solving differential equations without having to use integration! The LT method of solution is a most powerful method, which, as it happens, totally bypasses the need to perform any integration. Furthermore, the initial condition is "taken care of" at the outset - there is no need to determine separately any constant of integration. Hopefully, you can see that the LT method has "possibilities"!

Diagrammatically, the two processes can be considered as follows:



This doesn't look too big a deal, since the LT method used the unexplained variable, s, and needed the use of partial fractions. However, in reviewing the two solutions above you can see that in the LT method there is no need to use integration, the constant of integration is automatically found and there was no messy rearrangement of exponential equations required.

Add to these advantages that Laplace Transforms are extensively used in electronic, electrical and control systems, and it is clear to see that knowledge of how they work becomes imperative to any engineer.

Appendix

Analysis of different parts of the solution

An LR Series Circuit is a 'linear system', that is, it behaves according to the rule "what goes in, comes out". So a sinusoidal voltage as input, for example, results in a sinusoidal current as output. However, before the circuit settles down to a ***steady state*** sinusoidal output, it usually exhibits an exponential response (the ***transient*** – or 'short lived' - part of the response).

The LR differential equation's solution comes in two parts:

$$i = Ae^{-\frac{Rt}{L}} + f(t)$$

The **first part**, $Ae^{-\frac{Rt}{L}}$ is called the **Complementary Function** (CF) and, containing R and L, results from the system (circuit) itself. Here, $-R/L$ is always negative (since $R > 0$ and $L > 0$), so the CF is *always* exponential decay for non-trivial cases. A can be negative or positive; if it is positive, exponential decay occurs, if it is negative then exponential growth to a limit occurs. (For further information see/use the 'Exponential Function' applet from ***MathinSite***.)

Since $Ae^{-\frac{Rt}{L}}$ eventually decays away, this part of the solution is called the ***transient***.

The **second part**, $f(t)$ is called the **Particular Integral** (PI) and results from, and takes the same form as, the applied voltage (the input to the system). So, if the applied voltage is a sinusoid, the PI will also be a sinusoid (possibly a mix of sines **and** cosines); if the input is ae^{pt} then the PI is Be^{pt} (or $(Bt+C)e^{pt}$ under certain conditions).

$f(t)$ is the ***steady state*** part of the response.

The overall solution (current), $i(t)$, is the sum of the CF and PI.

Note that various mixes of values for the initial conditions, system parameters and applied voltage parameters ***can*** *result in any part of the full solution being zero.*

… and remember,
"the PI (steady state output) is the same form as the input"
is characteristic of ***linear*** systems such as the LR Series Circuit.