Mass-Spring-Damper Systems: 1
Unforced Systems Tutorial Worksheet

Learning Outcomes

- To develop, through experimentation, an understanding of how the mass-spring-damper system and its associated linear second-order differential equation respond to changing the system parameters and the initial conditions.
- To answer "what if ... ?" questions about unforced mass-spring-damper systems.

Introduction

The mass-spring-damper type of system and its associated second order differential equation occur in many areas of engineering; in particular, in various mechanical systems and electrical circuits (for the mathematical modelling and some background theory of mechanical systems, see the accompanying theory sheet, “Mass-Spring-Damper Systems – The Theory”).

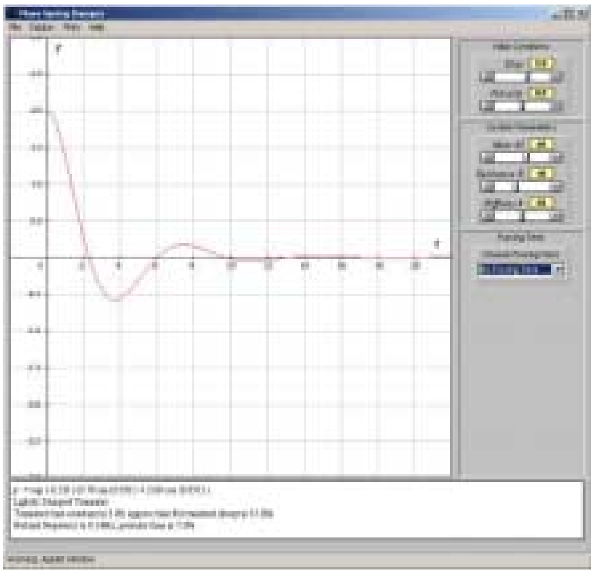
This tutorial worksheet and accompanying applet allow the user to investigate the response of an unforced mass-spring-damper system subjected to various initial conditions. The applet’s Windows-type slider bars are used to change the system’s parameters and the initial conditions.

To fully appreciate the working of this applet you should have already studied second order differential equations and their solution. The applet is not intended to teach you the subject but to give you a feel for how varying the parameters and initial conditions of a real world system affect its response.

Loading the Software

Select and run “Mass-Spring-Damper” from the choice of applets offered by the web site from which this document was downloaded.

Running the applet



When loaded, you see the applet window shown here. The applet loads with the default value of the mass’s initial displacement, *Disp* = 2.0, as shown by the top slider-bar on the right. No units are specified, so think in terms of all units being specified in SI units; metres, kilograms and seconds. (It is a BIG mass-spring-damper system!) The default value of initial velocity is zero (metres per second). The drop-down box on the right indicates that by default there is no forcing term, so the right hand side of the governing differential equation will be zero (see the accompanying theory sheet). The other three values loaded by default, as shown by their respective sliders, are M = 60, R = 40 and k = 50.

Note on Absolute and Relative Values
Bear in mind that the values M = 600, R = 400 and k = 500 would produce the SAME differential equation (when divided throughout by 10), so think in terms of the M, R and k values in relative terms as a ratio of $M : R : k$ rather than in terms of absolute values (e.g. M is not necessarily EQUAL to 60 kilograms - *unless specified so* - but is in the ratio of 60 : 40 : 50 when compared with R and k).

The corresponding system response is displayed in the main graphics area with system output (i.e. y, the displacement of the mass) on the vertical axis and time, t, on the horizontal axis. The analytical solution, and other information, is shown in the text area below.

The box halfway down the right hand side allows the user to change the type of forcing term (however only 'No forcing term' is considered in this, the first tutorial worksheet for this topic). The slider bars on the right of the applet window allow the user to change the value of system parameters and initial conditions. Clicking the arrowheads at either end of the slider bar effect *small changes* in the parameter value. Clicking in the main part of the box of the slider bar makes *larger changes*. By clicking, holding and moving the slider itself, you can make any sort of change.

Changing values by moving sliders is the only means of user-input.

The software gives you the option of showing single plots or multiple plots (under 'Plots' in the pull down menu structure at the top of the applet window). To see areas not covered by the default window, you can 'drag' the plot around by holding down the mouse button whilst moving the mouse over the plot. The plot can be re-centred at any time using the 'Centre on Origin' option, also under 'Plots'.

NOTE: This applet accommodates both mass-spring and mass-spring-damper systems. The applet does not allow zero values for mass, M, or spring stiffness, k. *Every* system will always need both a mass and a spring, but not all systems considered will have a resistance to the motion (so R *can* be zero). If k were allowed to be zero, i.e. no spring, then the mass would be a projectile. This is covered in the "Projectiles" applet (although, for completeness, may be incorporated here at a later date).

The Tutorial

The Unforced Mass-Spring-Damper System – Changing System Parameters

- ***Question***: Write down here the differential equation relating to the default parameter values.

Answer (provided this time!): $60\frac{d^2y}{dt^2} + 40\frac{dy}{dt} + 50y = 0$

[This, of course, could be written as $6\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 0$. Remember $M : R : k$]

- ***Question***: Describe the mass motion of the in this, the default mass-spring-damper system.

Answer (this time!): The mass is raised a distance of 2 metres and released from rest.

  - The *graphical solution* shows that there will be about 2 complete oscillations of the mass, but that these will die away within about 15 – 16 seconds when the mass returns to its datum position, $y = 0$ (its steady state value) – this sort of motion is characteristic of a lightly damped system.
  - The *analytical solution* shows sinusoidal terms modulated (multiplied) by exponential decay. The time constant for the exponential is 1/0.33, or approximately 3. Five times the time constant is about 15 as expected from the graph. The angular velocity of the oscillations was 0.85 rads/sec. This

gives a periodic time for the oscillations of $T = 2\pi/0.85 = 7.4$ seconds approximately. So in the 15 seconds taken to 'kill off' the oscillations there will be approximately 15/7.4 = 2 complete oscillations. The analytical solution corroborates the graphical solution showing that the transient part of the solution is decaying oscillations lasting for about 15 seconds and when these have 'died out' the steady state will be $y = 0$.

- The *parameter values* of the system show $R^2 - 4Mk = 40^2 - 4 \times 60 \times 50 = 1600 - 12000 < 0$ and $R^2 - 4Mk < 0$ is the condition that indicates that the system is lightly damped.

> Phew! There was a great deal of material here derived both from the graphics and the analytical solution. If you are unsure of any of the foregoing, it is recommended that you take time out to read through *at least* the sections on 'Time Constants and the Time to Decay' and 'Transients with Exponentially Decaying Sinusoids' in the accompanying theory sheet.

- Change the value of M to 8 whilst leaving R and k the same. Write down here the governing differential equation in this case.

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- Find the value of $R^2 - 4Mk$. What does this indicate?

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- From the graph, after approximately how long does it take to reach steady state?

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- Using the analytical solution shown in the text area, determine an appropriate "five times the time constant" for this system.

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- Keep a record of this case by sketching the resulting response curve, noting also the analytical solution and the list of parameter values. Label this as ***Figure 1, "Heavy Damping"*** so that you can reference it to these notes.

- Now change the value of M to 4 whilst leaving R and k the same. Write down here the governing differential equation in this case.

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- Find the value of $R^2 - 4Mk$. What does this indicate?

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- From the graph, after approximately how long does it take to reach steady state?

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- Using the analytical solution shown in the text area, determine an appropriate "five times the time constant" for this system.

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- These last two values don't really agree. Why not? From the analytical solution, $y = (2 + 5t)e^{-2.5t}$, determine the value of $y(2)$. What prevents the graphical solution from agreeing with the analytical solution?

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[*Moral: 5τ is only ever an approximation, and is a worse approximation when the exponential decay modulates (multiplies) an increasing function.*]

- Keep a record of this case by sketching the resulting response curve, noting also the analytical solution and the list of parameter values. Label this as ***Figure 2, "Critical Damping"*** so that you can reference it to these notes.

- In order to understand more intuitively the effect of changing parameter *M*, press and hold down the arrowhead at either end of the *M* slider bar. As *M* changes, so the system will pass from light damping, through critical damping to heavy damping and back again. For a longer-lasting visualisation of this effect, choose "Multiple Plots" from the "Plots" menu and repeat the process. Cool, huh?

- Of course, what you are visualising here are the three different types of damping (possible solutions) as the relative values of R^2 and $4Mk$ change. You can see the same effect (although on different timescale) if you now keep $M = 4$ and $k = 50$ and move the *R* slider. Remember, as you increase *R* (or decrease *M* or *k*) you are increasing the relative size of *R* (hence the damping) with respect to the mass and spring stiffness. So notice how much slower the system is by the time you have moved to $R = 100$ and how fast and springy it is when $R = 4$.

The Unforced Mass-Spring-Damper System – Changing the Initial Conditions

- Reset the default values: $M = 60$, $R = 40$, $k = 50$, $Disp = 2$ and $Velocity = 0$.
- Now change the value of *Disp* through 1, 0, -1, -2 by pressing inside the "*Disp*" slider bar 4 times to the right of the slider and noting the effect on the graph.
- How do these changes relate to the real-world system?

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- What did these changes do to the graph?

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- So, in real world terms, what does a value of *Disp* = ***minus*** 2 mean?

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- Did you notice the interesting result when $Disp = 0$? Describe graphically what you see for this case.

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- Explain in real world terms why you would expect this output.

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- Reset the default values: $M = 60$, $R = 40$, $k = 50$, $Disp = 2$ and $Velocity = 0$.
- Now change to *Disp* and *Velocity* to 0 and 1 respectively.
- Describe what these initial conditions mean and how the system responds.

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- Now click twice in the velocity box to the left of the slider to obtain $Velocity = -1$. Describe what difference this makes to the system response.

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- Bearing in mind that velocity = dy/dt, how could you represent the initial velocity on the plot of the system output?

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- Keep a record of these two cases by sketching the resulting response curves, noting also the analytical solutions and the list of parameter values. Label this as ***Figure 3*** so that you can reference it to these notes.

- Now set the values: M = 60, R = 10, k = 50, *Disp* = 2 and *Velocity* = 0.
- State the type of damping and describe how the system responds.

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- Click repeatedly the arrowhead at the left hand end of the R slider so R changes to 9, 8, 7, 6, 5, 4. Watch the graphs. Stop at 4. Describe what you are doing to the system in real world terms and what effect it is having on the system response.

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- Now change R to 3, 2, 1 and finally zero. In real world terms what have you done to the system when R = 0?

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- What is the value of $R^2 - 4Mk$? What type of damping is present in the system?

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- What type of response do you see when R = 0?

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- This is called ***Simple Harmonic Motion*** (SHM).

Exercises

1. Set up *any* system, but one in which both initial conditions are zero and note the response. Surprised? Explain why you get such a response.

2. Use the applet to set up *any* unforced system that is *heavily damped* with initial conditions both not zero. Determine the time constants for both of the exponential terms and hence verify the time to steady state indicated in the lower part of the applet window. Do the graphics show this transient time fairly accurately? If so, why? If not, why not? Write down the parameters that relate to your system, sketch the graphical output and write down the analytical solution.

3. Use the slider bars in the applet to design *any* unforced system that has initial conditions *Disp* = 1 and *Velocity* = -1 and an oscillating transient that lasts for about 5 seconds before reaching steady state. Write down the parameters that relate to your system, sketch the graphical output and write down the analytical solution.

4. A unforced second order mechanical system is to be designed to be critically damped with $M : R : k$ in the ratio of 4 : R : 9. Use the sliders to set up M = 4 and k = 9 and move the slider bar for R until the analytical (and graphical) solution shows a critically damped system. Use any non-zero initial conditions. Check the value of $R^2 - 4Mk$ to verify that this gives a critically damped system. Determine the time constant from the analytical solution shown and hence the approximate time to steady state. Write down the parameters that relate to your system, sketch the graphical output and write down the analytical solution.

5. Repeat the work you did in the previous question with the same initial conditions, but this time using *double* the values used for *M*, *R* and *k* (i.e. 8, 2*R*, and 18, using the value of *R* found in Question 3). What difference did it make to your answers? Why?