Name ………………………….………. Course ……………………………………….
Print and use these sheets with the Mass-Spring-Damper applet – giving you a permanent record of your work.

Mass-Spring-Damper Systems: 2
Sinusoidally Forced Systems
Tutorial Worksheet

Learning Outcomes

- To develop, through experimentation, an understanding of how the mass-spring-damper system and its associated linear second-order differential equation responds to changing the system parameters, the amplitude and frequency of an applied sinusoidal force and the initial conditions.
- To answer "what if ... ?" questions about mass-spring-damper systems.

Introduction

The mass-spring-damper type of system and its associated second order differential equation occur in many areas of engineering; in particular, in various mechanical systems and electrical circuits (for the mathematical modelling and some background theory of mechanical systems, see the accompanying theory sheet, "Mass-Spring-Damper Systems – The Theory").

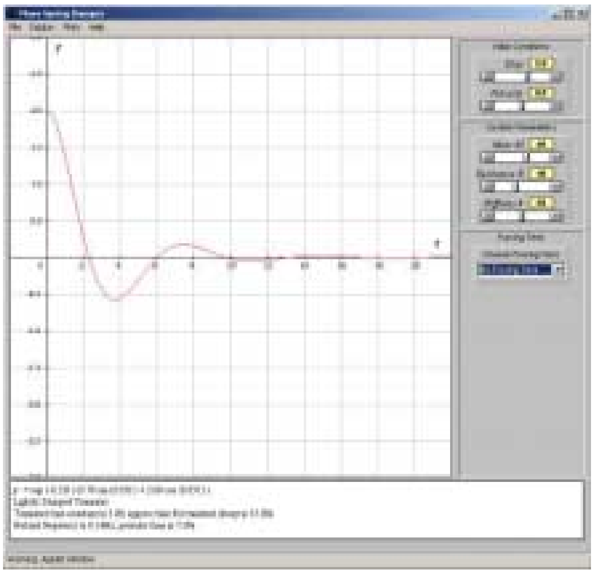
This tutorial worksheet and accompanying applet allow the user to investigate the response of mass-spring-damper systems subjected to a sinusoidal driving force with zero or non-zero initial conditions. This is achieved using Windows-type slider bars to change the system and external force's parameters and the initial conditions.

To fully appreciate the working of this applet you should have already studied second order differential equations and their solution. The applet is not intended to teach you the subject but to give you a feel for how varying the parameters, forcing terms and initial conditions of a real world system affect its response.

Loading the Software

Select and run "Mass-Spring-Damper" from the choice of applets offered by the web site from which this document was downloaded.

Running the applet



When loaded, you see the applet window shown here. The applet loads with the default value of the mass's initial displacement, *Disp* = 2, as shown by the top slider-bar on the right. No units are specified – but since the value of *g*, the acceleration due to gravity, is taken to be 9.81 in the applet's calculations, all units will be SI units in terms of metres, kilograms and seconds. (It is a BIG mass-spring-damper system!) The default value of initial velocity is zero (metres per second). The drop-down box on the right indicates that by default there is no forcing term, so the right hand side of the governing differential equation will be zero (see the accompanying theory sheet). The other three values loaded by default are $M = 60$, $R = 40$ and $k = 50$, as shown by their respective sliders.

Note on Absolute and Relative Values
Note that values of *M* = 600, *R* = 400 and *k* = 500 would produce the SAME differential equation (when divided throughout by 10), so think in terms of the *M*, *R* and *k* values in relative terms as a ratio of *M* : *R* : *k* rather than in terms of absolute values (e.g. *M* is not necessarily EQUAL to 60 kilograms - *unless particularly specified as such* - but is in the ratio of 60 : 40 : 50 when compared with *R* and *k*).

The corresponding system response is displayed in the main graphics area with system output (displacement of the mass) on the vertical axis and time on the horizontal axis. The analytical solution, and other information, is shown in the text area below.

The box halfway down the right hand side allows the user to change the type of forcing term. The slider bars on the right of the applet window allow the user to change the value of system parameters. Clicking the arrowheads at either end of the slider bar effect *small changes* in the parameter value. Clicking in the main part of the box of the slider bar makes *larger changes*. By clicking, holding and moving the slider itself, you can make any sort of change.

Changing values by moving sliders and choosing forcing terms from the drop-down box is the only means of user-input.

The software gives you the option of showing single plots or multiple plots (under 'Plots' in the pull down menu structure at the top of the applet window). To see areas not covered by the default window, you can 'drag' the plot around by holding down the mouse button whilst moving the mouse over the plot. The plot can be re-centred at any time using the 'Centre on Origin' option, also under 'Plots'.

The Tutorial

(Note: It is assumed that you have already completed the first tutorial sheet, on "Unforced Systems")

Forced Mass-Spring-Damper Systems – Changing System Parameters

- From the "Choose Forcing Term" drop-down box on the right, choose "Sinusoidal Forcing Term". What effect did this have on the response on the output for the unforced system that was originally in the graphics area? (Go back to "No forcing ter" and then "Sinusoidal Forcing Term" again if you missed it the first time!)

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- In particular, what happened to the original transient response?

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- This happens because the amplitude of the sinusoidal terms is very small compared with the transient response. Click and hold the "*a*" slider in the first slider bar below the drop-down box and move it slowly backwards and forwards to the right to see the effect of increasing the forcing term's amplitude. Release it at an "*a*" value of 100. For a larger effect, increase "*b*" to 100 as well.
- Describe the effect that increasing this amplitude has on the overall response.

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- What type of response is the steady state solution in this case? (Remember that this is a *linear* system – what goes in, comes out - and that 'steady' in 'steady state' doesn't necessarily mean 'constant')

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- Note that this steady state response is directly superimposed on to the transient as well as itself being the steady state solution (when the transient has died away).

- ***Question***: Write down here the differential equation relating to these parameter values.

Answer (provided this time!): $60\frac{d^2y}{dt^2} + 40\frac{dy}{dt} + 50y = 100\sin 3t + 100\cos 3t$

[or, $6\frac{d^2y}{dt^2} + 4\frac{dy}{dt} + 5y = 10\sin 3t + 10\cos 3t$, giving the same system]

- ***Question***: Describe the motion of the default mass-spring-damper system.

Answer (provided this time!):

- ➢ The mass is raised a distance of 2 metres and released from rest.
- ➢ The graph shows a lightly damped transient (from the unforced system, already covered in the first tutorial sheet) with about 2 complete oscillations decaying within about 15 – 16 seconds.
- ➢ After the transient has decayed, the mass undertakes steady state oscillations, with steady state displacement, y_{ss}, given by $y_{ss} = -0.15\sin 3t - 0.24\cos 3t$ (from the solution shown in the lower text area).
- ➢ These steady state oscillations can be seen to be operating during the transient time also - big oscillations with little oscillations on them. Linear systems such as these superimpose steady state solution and transient, *for all time* (except that the transient will, eventually, die away).
- ➢ The angular velocity of the transient oscillations is 0.85 rads/sec giving a periodic time of $T = 2\pi/0.85$, about 7.4 seconds. The angular velocity of the forcing oscillations is 3 rads/sec giving a periodic time of $T = 2\pi/3 = 2.1$ seconds approximately. The steady state part of the solution manages just over three oscillations for each of the transient's oscillations.

Phew! There was a great deal of material here derived both from the analytical and the graphical solution. You should have seen this sort of discussion in the first tutorial sheet, but if you're still not sure, take time out to read through *at least* the sections on 'Time Constants and the Time to Decay' and 'Transients with Exponentially Decaying Sinusoids' in the accompanying theory sheet.

- Keep a record of this case by sketching one the resulting response curves you obtained, noting also its analytical solution and the list of parameter values. Label this as ***Figure 1, "Light Damping with sinusoidal forcing"*** so you can reference it to these notes.
- Now *increase* the value of ω by small amounts by clicking on the arrowhead on the right of the ω slider bar. What is the effect on the overall response when increasing the value of ω?

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- What effect did this have on the transient response?

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- Now use the arrowhead on the left end of the ω slider to gradually reduce its value until you reach about $\omega = 0.9$. What effect did this have on the response? (You may use the word "bananas" in you answer!)

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- What can you say about the frequency (= $1/T$) of the applied oscillations and those due to the system itself (as shown in the transient)?

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- Describe in real world terms what you are doing to the system (in particular when compared with the system's own oscillations).

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- The transient oscillations are being swamped by the steady state oscillations. In a mechanical system, this can be catastrophic especially in very lightly damped or undamped (with $R = 0$) systems - as mentioned in the accompanying theory sheet. This is important enough for you to be told here that if you drive a lightly damped system by an oscillating force with a frequency close to the system's natural frequency the system response oscillations are reinforced. This can be seen much more readily in the following examples.

- Use the slider bars to set the following values: *Disp* = 0, *Velocity* = 0, $M = 10$, $R = 1$, $k = 90$ and a sinusoidal forcing term with $a = 20$, $b = 0$ and $\omega = 10$. Write down here the differential equation relating to the values.

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- Write down the analytical solution here.

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- You should see a very lightly damped system with $R^2 << 4Mk$ so the transient response is almost simple harmonic motion (SHM). Superimposed on these oscillations are the oscillations due to the forcing term.
- From the analytical solution determine the frequency in Hz of the natural and forced oscillations.

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- Now press and hold down the arrowhead at the left hand end of the ω slider and watch as ω reduces from 10 to about 3.7. Stop here. Now step down through 3.6, 3.5 to 3.4. At ω =3.4 describe what you see in the response (you may need to scroll the graphics window to the right to see this effect more clearly).

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- Keep a record of this case by sketching the resulting response curve, noting also the analytical solution and the list of parameter values, etc. Label this as ***Figure 2***, ***"Beats"*** so that you can reference it to these notes and cross-reference it with the discussion on Beats in the Theory Sheet.

- Scroll the graph back so you can see the origin.

- Now take out all the resistance to motion by setting $R = 0$. Notice that this slightly increases the amplitude of the response oscillations.
- By clicking the arrowhead on the left of the ω slider, step down from $\omega = 3.4$ one step at a time through 3.3, 3.2, 3.1. Stop there. What happened to the beats? Again you may need to scroll the graph to the right.

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- Scroll the graph back so you can see the origin. Now step down to 3.0. Scroll the graph again so that you can see where it goes to the right and upwards. As you moved ω from 3.5 down to 3.1 the wavelength of the beats increased. When you hit 3.0 (and with nothing in the system to resist any motion, since $R = 0$) the wavelength of the beats becomes infinite – the system looks as though it's about to perform beats but never reduces the amplitude of the oscillations. In theory the amplitude of the oscillations increase without limit. However, in real world terms this is not possible and the system "breaks" – to put it mildly!

- The analytical solution shows that the sinusoid (in the 'so-called' steady state) is modulated by a linear term, i.e. the oscillations are multiplied by the term *t*, the amplitude of which increases without limit as time inexorably marches on.
- Keep a record of this case by sketching the resulting response curve, noting also the analytical solution and the list of parameter values. Label this as ***Figure 3,*** "***Resonance***" so that you can reference it to these notes and cross-reference this with the discussion on Resonance in the Theory Sheet.

- True resonance only occurs in undamped systems (although you can get pretty close for very small *R*). Increase the value of *R* by holding down the arrowhead at the right hand end of the *R* slider (i.e. increase the system's resistance to motion). The solution now shows a "Lightly Damped System" as you do this and you can see that the solution soon settles down to a steady state set of pure oscillations with finite amplitude. Increasing *R* beyond about 60 changes the transient to one that is heavily damped.

Exercises

1. There are three ways in which you can help solve the problems of near resonance at the Millennium Bridge in London (see accompanying Theory Sheet). You now know that light damping can lead to problems in mechanical systems; so one solution would be to increase the resistance to motion. What else can you do to eliminate such problems? [Hint: think $R^2 - 4Mk$]. Discuss how any of this could be done in real world terms and send your solution to the Millennium Bridge Commission, London!

2. A second order linear system of the mass-spring-damper type has system parameters $M = 1$, $R = 4$ and $k = 1$. The initial conditions are *Disp* = 2 and *Velocity* = 0 and the system is excited by an oscillating force $F(t) = \sin 4t$. Enter this data into the applet and use either the analytical result or the graph to estimate the transient time. Although the manufacturer requires steady state oscillations with angular velocity 4 rad/sec, the transient response is not fast enough. Change the value of the resistance factor, *R*, until critical damping is achieved. Sketch the resulting response curve, noting also the list of parameter values and the analytical solution.

3. A mass-spring-damper system is to be driven sinusoidally. The system parameters are such that $M : R : k : a : b = 20 : 10 : 80 : 0 :20$, the angular frequency of the oscillations is 2 rad/sec and the initial conditions are both zero. Enter these values into the applet and write down the differential equation relating to this system. Is there a more simplified form of what you have just written? (Remember that $M : R : k : a$: b refers to *relative* values, not *absolute* values.) The manufacturer considers the system is too responsive and is willing only to change the spring in order to achieve a maximum amplitude anywhere in the motion of 0.5 metres. Use the applet to determine the maximum value of *k* in $M : R : k : a : b = 20 : 10 : k : 0 :20$ that satisfies this requirement.

4. An undamped system with an applied oscillating force of angular velocity 3 rad/sec has $M : R : k : a : b = 1 : 0 : 36 : 0 : 30$ and zero initial conditions. Write down a differential equation relating to this data. Determine the periodic time of the "transient" SHM of the system and the periodic time for the applied oscillations. What do you notice, in particular explain why you obtain the periodic output as shown? Why was the word "transient", above, in quotes? (Use of "British sense of irony" is acceptable in your answer.) What do you think would happen if you changed $M : R : k : a : b$ to $1 : 0 : 81 : 0 : 30$ and kept an angular velocity of 3 rad/sec for the applied oscillating force? [Hint: do the same as for the first case.] Use the applet to corroborate your answer.