Name ……………………….……….   Course ……………………………………….

Mass-Spring-Damper Systems: 3
Exponentially and
Algebraically Forced Systems
Tutorial Worksheet

Learning Outcomes

- To develop, through experimentation, an understanding of how the mass-spring-damper system and its associated linear second-order differential equation responds to changing the system parameters, the applied forces (in this case, algebraic and exponential forcing) and the initial conditions.
- To answer "what if ... ?" questions about mass-spring-damper systems.

Introduction

By the time you get to this worksheet you should have completed worksheets 1 and 2. In 1, the mass-spring-damper system was unforced. In 2, an applied sinusoidal force influenced the system's response. In this worksheet, you investigate what happens when an exponentially or algebraically varying force is applied. Under what circumstances are such forces likely to occur? If *any* external force applied to a system is removed, its effect may not disappear instantaneously, for example. The magnitude of this force may be modelled by a linear or a quadratic decrease, or exponential decay.

This tutorial worksheet and accompanying applet allow the user to investigate the response of mass-spring-damper systems subjected to such forces with zero or non-zero initial conditions. This is achieved using Windows-type slider bars to change the system and external force parameters and the initial conditions.

To fully appreciate the working of this applet you should have already studied second order differential equations and their solution. The applet is not intended to teach you the subject but to give you a feel for how varying system parameters, forcing terms and initial conditions of a real world system affect its response.

Loading the Software

Select and run "Mass-Spring-Damper" from the choice of applets offered by the web site from which this document was downloaded.

Running the applet

When loaded, you will see the applet window shown here. The default values and the units used in the applet have already been discussed in the previous worksheets. The drop-down box on the right indicates that by default there is no forcing term. Clicking in this box or on the drop-down arrow enable you to choose exponentially or algebraically varying applied forces.

The corresponding system response is displayed in the main graphics area

---

Print and use these sheets with the Mass-Spring-Damper applet – giving you a permanent record of your work.
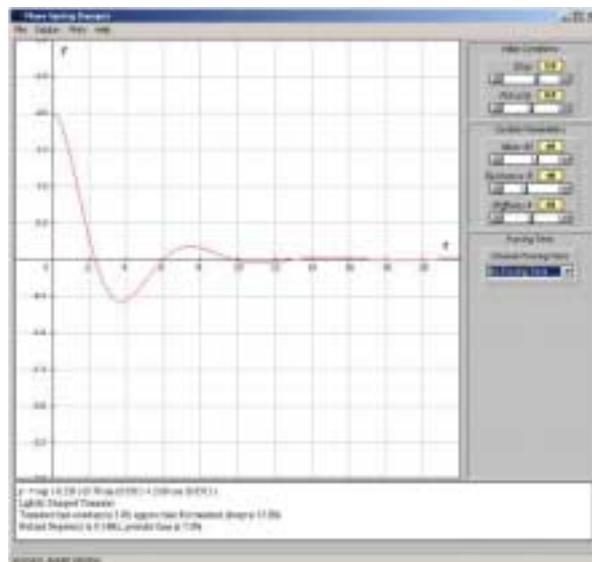
with system output (displacement of the mass) on the vertical axis and time on the horizontal axis. The analytical solution, and other information, is shown in the text area below.

The slider bars on the right of the applet window allow the user to change parameter values. Clicking the arrowheads at either end of the slider bar effect *small* changes, clicking in the main part of the box of the slider bar makes *larger changes*. By clicking, holding and moving the slider itself, you can make any sort of change.

Changing values by moving sliders and choosing forcing terms from the drop-down box is the only means of user-input.

The software gives you the option of showing single plots or multiple plots (under 'Plots' in the pull down menu structure at the top of the applet window). To see areas not covered by the default window, you can 'drag' the plot around by holding down the mouse button whilst moving the mouse over the plot. The plot can be re-centred at any time using the 'Centre on Origin' option, also under 'Plots'.

The Tutorial
(Note: It is assumed that you have already completed the first two tutorial sheets)

Forced Mass-Spring-Damper Systems – Exponentially Varying Applied Forces
- The software when run will load with its default values, $M = 60$, $R = 40$, $k = 50$, *Disp* $= 2$ and *Velocity* $= 0$
- From the "Choose Forcing Term" drop-down box choose "Exponential Forcing Term". This loads with the default values $a = 15$ and $b = -1.5$.
- What difference does driving the system with this force have on the output compared with the unforced system response that was originally in the graphics area? (Go back to "No forcing term" and then "Exponential Forcing Term" again if you missed it the first time!)

…………………………………………………………………
- Remember that linear systems such as these are such that "what you put in, you get out". If you drive a system with an exponential force, the 'steady state' part of the response will also be exponential – with the same exponent.
- Why does this driving force have so little effect on the response? To help answer this it may help if you calculate five times the 'time constant' (5τ) for the exponential terms both in the transient and in the steady state parts of the response (or see the lower part of the applet's text window).

…………………………………………………………………
- Effectively the steady state dies away before the transient! What if it doesn't? Click once in the "b" slider in the second slider bar below the drop-down box to obtain a b value of -0.5. What is 5τ for this driving force (and hence steady state part of the solution)? Compare it with the transient's 5τ. Was this change visible?

…………………………………………………………………
- You're going to have to increase b beyond -0.3 (why?) to see any effect. Use the b slider to obtain $b = -0.2$. Even so this is difficult to see!
- The problem is that the transient and steady state are taking about the same time to decay. But now move the b value just the other side of zero, to $b = +0.1$. The exponential driving force no longer decays; $e^{0.1t}$ is exponential growth! An

exponentially growing applied force means an exponentially growing 'steady state'.

- You can see the effect of this applied force quite well if you now toggle between "No Forcing Term" and "Exponential Forcing Term" – your values of *a* and *b* are maintained as you do this. Note how the underlying steady state is too small to be noticed during the first 2 seconds, but after that, how the exponential growth takes over! (Toggling with "Multiple Plots" on shows this effect clearly.)

- On a separate piece of paper, write down the differential equation and the analytical solution relating to this set of values. Sketch the full response curve that the applet displays. Superimpose, as a dotted line, the system's response curve without the forcing. Label both graphs to distinguish between the two cases. Label the whole diagram as *Figure 1, "Exponentially Forced Mass-Spring-Damper"* so that you can reference it to these notes.

- Now set "Single Plots" and use the sliders to set: *Disp* = 2.0, *Velocity* = 0, *M* = 10, *R* = 30, *k* = 20 and an exponential forcing term with *a* = 5 and *b* = -2.3. Write down the differential equation relating to the values and the analytical solution. Write down 5τ for both the system and the forced part of the solution. Theoretically, it should not be possible on the graph to see the effect of the steady state. Why not? (Again you can toggle with "Multiple Plots" between "No forcing" and "Exponential forcing" to see this effect more clearly.)

…………………………………………………………………

- However, if you move the "*a*" slider, which represents the magnitude of the forcing term and hence is related to the magnitude of the steady state, you can see that it does affect the response before it disappears (i.e. before about 2.5 seconds). Try it now by moving the "*a*" slider to a value of 100.

- Write down the analytical solution for this case here.

…………………………………………………………………

- Now click repeatedly inside the "*b*" slider track to obtain values of *b* = -2.2, -2.1, -2.0, -1.9. Write down the analytical solution in each case:

b = -2.2, …………………………………………………

b = -2.1, …………………………………………………

b = -2.0, …………………………………………………

b = -1.9, …………………………………………………

- One of these solutions had a slightly different form to the others (I hope you noticed!). What relationship did this case have with the transient solution?

…………………………………………………………………

NOTE: When you force a system with a term that already appears in the transient part of the response, the "usual" steady state type of response is always modulated by the dependent variable (*t* in this applet). In the above, *b* = -2.0 was equivalent to the sinusoidally forced case (see Tut Sheet 2) when the system was undamped, in that case it lead to problems involving resonance. Here it has no significant effect.

- There will be another value of *b* for this case that will produce the same effect. Move the "*b*" slider until you reach this value and write down here the value of *b* and the system's analytical solution.

b =   , …………………………………………………

Print and use these sheets with the Mass-Spring-Damper applet – giving you a permanent record of your work.

- Keep a record of this case by sketching the resulting response curve, noting also the analytical solution and the list of parameter values. Label this as **Figure 2** so that you can reference it to these notes. Label your graph "***Forcing with exponentials that appear in the transient***".

NOTE: Most of the above cases use exponential **decay**. The software only allows you to enter at worst exponential **growth** of $e^{0.1t}$. Exponentials with b greater than 0.1 become large fairly quickly. At $t = 1$, e^{10t}, for example, is approximately 22,026 and at $t = 10$, it is about 2.7 x 10^{43}. A bit off the scale!

Forced Mass-Spring-Damper Systems – Algebraically Varying Applied Forces

- Reset the default values: $Disp = 2$ and $Velocity = 0$, $M = 60$, $R = 40$, $k = 50$
- You may have moved the graph, therefore use "Centre on Origin" from the "Plots" menu to reset the graph so the origin can be seen.
- From the "Choose Forcing Term" drop-down box choose "Algebraic Forcing Term". If you have just loaded the applet anew, this loads with the default values $a = 1.5$, $b = -2.5$ and $c = 5.0$. If you are continuing the applet from the previous exponential case, you will need to set these values for a, b and c.
- What difference does driving the system with this force have on the output compared with the unforced system response that was originally in the graphics area? (Go back to "No forcing term" and then "Algebraic Forcing Term" again if you missed it the first time!)

……………………………………………………………………

- Remember that linear systems such as these are such that, "What you put in, you get out". If you drive a system with an algebraic (in this case, more specifically, a quadratic) force, the steady state part of the response will also be quadratic – graphically, a parabola. The response parabola, usually, will not be the same as the forcing parabola. Write down here the forcing term, F_{ft}, and the corresponding steady state output, y_{ss}, and notice their similarity.

$F_{ft} =$ ……………………… $y_{ss} =$ ……………………

- After about how long does the response curve become the steady state parabola? (Remember 5τ.)

……………………………………………………………………

- On a separate piece of paper, write down the differential equation and the analytical solution for this case. Sketch the response curve. Label this as **Figure 3, "Algebraic (Quadratic) Forcing Term"** so that you can reference it to these notes. Extend the steady state parabola back as a dotted line to the y-axis (its equation is $y =$ "the steady state (quadratic part) of the analytical solution").

- Change a to zero. What do you notice now about the steady state part of the response? (Remember, "What you put in, you get out".) You may need to "grab-hold-and-pull" the graph to the right to see this more clearly (and then move it back again).

……………………………………………………………………

- Now change b to zero as well, but before you do, can you guess what the steady state will look like this time? Try it now. Write down here the forcing term, F_{ft}, and the corresponding steady state output, y_{ss} and notice their similarity.

$F_{ft} =$ ……………………… $y_{ss} =$ ……………………

Name ……………………….……….   Course ……………………………………….

NOTE: If you force a mass-spring-damper system with a constant force, so that the governing differential equation is $My'' + Ry' + ky = c$, then the steady state constant output always takes the value c/k. You may like to check this out when you have finished the tutorial.

- This time you should have no problem. If you were to change c to zero as well, what case of forcing term would this represent?

………………………….………………………………………

- You can see the effect of this applied force quite well if you now toggle between "No Forcing Term" and "Algebraic Forcing Term".

Exercises

1. Set up a *lightly* damped system with constant input and the default initial conditions such that the steady state value is 3.2 and the time for the oscillations to disappear is approximately 5 seconds. What is the periodic time of the transient oscillations in your system? Write down the differential equation and the analytical solution relating to this set of values. Sketch the response curve that the applet displays. Label this as ***Figure 4, "Constant Forcing Term"*** so that you can reference it to these notes.

2. In Q.1, what difference does it make to the response if you change the initial conditions? Why don't these changes affect the steady state solution?

3. A mass-spring-damper system has parameter values $M = 1$, $R = 2$ and $k = 1$. The initial conditions are $Disp = 2$ and $Velocity = 0$ and the system is excited by an exponential force $F_{ft} = e^{-t}$. In the above text, it was mentioned that if the forcing term appears in the transient, then the steady state solution is modulated (multiplied) by t. In this solution it is multiplied by t^2. Why do you think this happens in this case? Write down the differential equation and the analytical solution relating to this set of values. Sketch the response curve that the applet displays. Label this as ***Figure 5, "Forcing with Exponentials that …"*** (where you use your own words after the "that"), so that you can reference it to these notes.

4. A mass-spring-damper system is to be driven by a ramp force (one that varies linearly with time). The system and driving force are defined by values $M = 20$, $R = 10$, $k = 80$, $a = 0$, $b = 0.5$ and $c = 3.0$, (i.e. the ramp force is $F_{ft} = 0.5t + 3.0$) and the initial conditions are both zero. Enter these values into the applet. The manufacturer requires that the ramp response reaches $y = 3.0$ at about $t = 30$ seconds. Use the applet to determine the maximum value of b (keeping c the same value) that achieves this requirement. Write down the differential equation and the analytical solution relating to this set of values. Sketch the response curve that the applet displays. Label this as ***Figure 7, "Ramp Forcing Term"*** so that you can reference it to these notes.

5. A heavily damped, exponentially forced mass-spring-damper system is to be set up with parameters in the ***ratio $M : R : k : a = 0.1 : 0.4 : 0.3 : 1$***. The forcing term is of the form $F_{ft} = Ae^{-0.1t}$. Enter this system into the applet with zero initial conditions. You will realise that you are not allowed to enter $M = 0.1$, etc. However, remember that $M : R : k : a$ is a ***ratio***. Use the applet to determine what difference is made to the response when the initial conditions are changed.