# Piecewise, Odd/Even and Periodic Functions Theory Sheet

## Learning Outcomes

After reading this theory sheet, you should be able to:

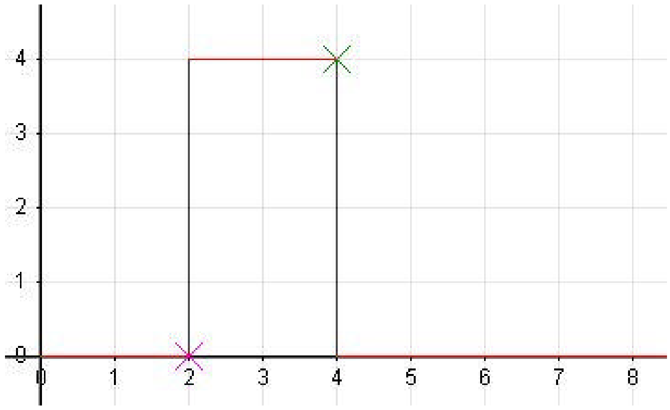
- Recognise, sketch the graphs of , and evaluate points on
    - piecewise defined functions
    - functions that are even, odd, or neither odd nor even
    - periodic functions

## Piecewise functions

Engineers use many basic mathematical functions to represent, say, the input/output of systems - linear, quadratic, exponential, sinusoidal and so on – but how are these used to generate some of the more unusual input/output signals such as the square wave, saw tooth wave and fully rectified sine wave?

Consider the following single pulse function, $f(x)$:

(This diagram is part of a screenshot from the accompanying ***MathinSite*** applet in which the movable crosshair cursors shown are used to describe the function graphically.)




This function is zero everywhere, except where it takes the value 4. It is described in different ways for different parts of its *domain*, (the domain is the range of values that can be taken by $x$). It is a ***piecewise function***, since it is defined in 'pieces'.

| A **piecewise defined function** is one which is defined using two *or more* formulas. |
|---|

Here, it is

$$f(x) = \begin{cases} 0 & x < 2 \\ 4 & 2 \le x < 4 \\ 0 & x \ge 4 \end{cases} \quad \text{or, alternatively,} \quad f(x) = \begin{cases} 0 & x \le 2 \\ 4 & 2 < x \le 4 \\ 0 & x > 4 \end{cases}$$

Note the difference between these two *equally valid* forms.

In the first, the function takes the value zero from minus infinity all the way up to, *but not including*, $x = 2$, then takes the value 4 *from* $x = 2$ up to, *but not including*, $x = 4$. From 4 inclusive onwards, the function returns to zero.

In the second, the function takes the value zero from minus infinity all the way up to, *and including*, $x = 2$, then takes the value 4 *from* $x = 2$, not included, up to, *and including*, $x = 4$. From 4 not included onwards, the function returns to zero.

Consider the following piecewise description for the previous pulse function.

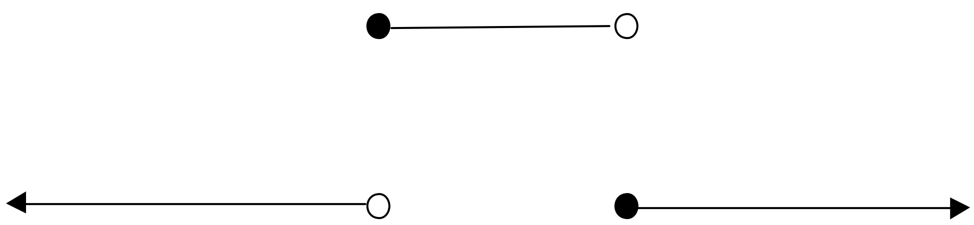
$$f(x) = \begin{cases} 0 & x \le 2 \\ 4 & 2 \le x \le 4 \\ 0 & x > 4 \end{cases}$$

This would not be allowed since at the 'joining' point $x = 2$ the function is defined in two different ways, the function takes both of the values 0 and 4 at $x = 2$. This contravenes the 'single input, single output' requirement of all functions that a single value of $x$ into the function should result in a single value out of the function.

In order to distinguish graphically which piece takes an end value, use is made of a hollow circle or a filled circle. The hollow circle indicates that the endpoint is not included and the filled circle indicates that it is.

So for the definition $f(x) = \begin{cases} 0 & x < 2 \\ 4 & 2 \le x < 4 \\ 0 & x \ge 4 \end{cases}$,

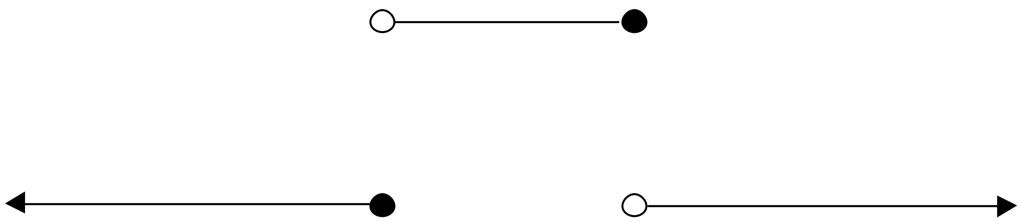
the above the graph (with the axes removed for clarity) should strictly look like:



$x = 2$ $x = 4$

The arrows indicate that the domain is not bounded to the left or the right – the whole domain stretches from minus infinity to plus infinity.

For the second definition, $f(x) = \begin{cases} 0 & x \le 2 \\ 4 & 2 < x \le 4 \\ 0 & x > 4 \end{cases}$, the graph would be



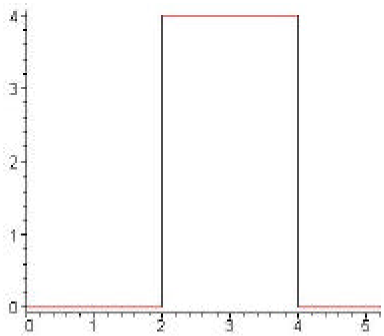
$x = 2$ $x = 4$

Note what else, apart from the axes, is missing from both of these graphs when compared with the ***MathinSite*** screenshot. *The vertical lines at $x = 2$ and $x = 4$ joining the endpoints of each piece*. So should they be there, or should they not? Mathematically speaking, the answer is

'No'. The function shown is an example of a ***discontinuous function***. At $x = 2$, the functions jumps instantaneously – there is a discontinuity – from the value $f(x) = 0$ to $f(x) = 4$. As the function approaches $x = 2$ from the left it takes the value of zero; whereas approaching $x = 2$ from the right it has the value 4. *The function is discontinuous at* $x = 2$ (and again, of course, at $x = 4$).

So why does ***MathinSite*** include the vertical line - and do other Mathematics packages do the same?

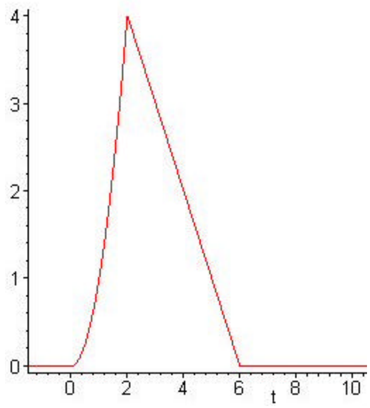
***MathinSite*** is primarily aimed towards engineering undergraduates where, for example, a pulse function is a standard 'signal' used in engineering systems. Such a signal can be visualised on an oscilloscope, where a beam electronically traces out the signal across the screen. For the above example, the beam would not switch off at $x = 2$ when tracing out the signal's instantaneous rise from zero to 4 and hence the beam traces out the vertical sides of the pulse – in fact, making it more pulse like.




Mathematics software packages such as *Maple* also show the 'sides' of such pulses. The diagram on the right is part of a *Maple* screenshot.

Piecewise functions can, of course, be ***continuous***. Consider the following function.

$$f(t) = \begin{cases} 0 & t < 0 \\ t^2 & 0 \le t < 2 \\ -t + 6 & 2 \le t < 6 \\ 0 & t \ge 6 \end{cases}$$




If a piecewise (non-rational) function is going to be discontinuous, it is only ever going to be discontinuous at the points where the function changes its definition. For this example, at $t = 0$, 2 and 6.

At $t = 0$ the function approaches a value of zero both from the left (where it is defined as 0) and from the right (where it is defined as $0^2$). In mathematical terms this can be written

$$\lim_{t \to 0^-} \{ f(t) \} = \lim_{t \to 0^+} \{ f(t) \} = 0$$

i.e. in the limit, as $t = 0$ is approached from the negative side, the function takes the same value as when $t = 0$ is approached from the positive side, here zero.

At $t = 2$ the function approaches a value of 4 both from the left ($2^2$) and from the right (-2 + 6), i.e. $\lim_{t \to 2^-} \{ f(t) \} = \lim_{t \to 2^+} \{ f(t) \} = 4$

At $t = 6$ the function approaches a value of zero both from the left (-6 + 6) and from the right (0), i.e. i.e. $\lim_{t \to 6^-} \{ f(t) \} = \lim_{t \to 6^+} \{ f(t) \} = 0$

Therefore the function is continuous for all points in the domain – as can be seen from the graph.

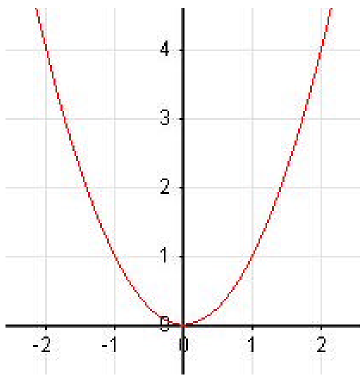
## Even and Odd Functions

All functions must be odd, even, or neither odd nor even. From a graphical inspection, it is fairly straightforward to determine in which category a function belongs.

**Graphical Definitions**

- $y = f(x)$ is an EVEN function if its graph
  - is symmetric with respect to the $y$-axis, i.e. it reflects in the $y$-axis.
- $y = f(x)$ is an ODD function if
  - its graph is symmetric with respect to the origin, i.e. the graph reflects through the origin, or
  - its graph rotates about the origin through $180^o$ without changing its form, or
  - the shape of the graph is unchanged if it is reflected in the $y$-axis, then the $x$-axis (or *vice versa*)
- **else**, $f(x)$ is neither ODD nor EVEN.

These ideas are best illustrated using some basic functions.

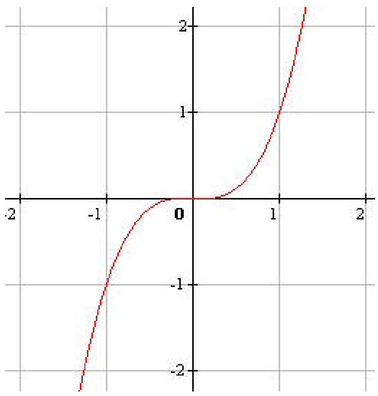



<u>The diagram shows the graph of $f(x) = x^2$.</u>

This graph can easily be seen to be symmetric about the $y$-axis – the graph reflects in the $y$-axis – so, $f(x) = x^2$ is an **EVEN** function.

Other functions that can be seen in a similar, graphical, way to be **even** functions are

- $f(x) = x^2 + 2$,
- $f(x) = \cos x$,
- $f(x) = 1/x^2$,
- $f(x) = x^4$, etc



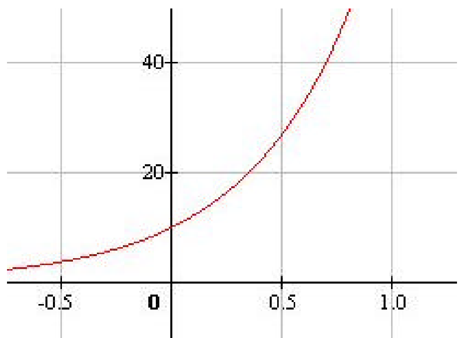

<u>The diagram shows the graph of $f(x) = x^3$.</u>

This graph can easily be seen to be symmetric about the *origin* – rotating the graph through $180^o$ about the origin, or reflecting the graph in the $y$-axis, then the $x$-axis (or *vice versa*) leaves the shape unchanged – so, $f(x) = x^3$ is an **ODD** function.

Other functions that can be seen in a similar, graphical, way to be **odd** functions are

- $f(x) = mx$, (yes, a straight line, slope $m$, through the origin)
- $f(x) = \sin x$,
- $f(x) = 1/x$, (rectangular hyperbola)
- $f(x) = \tan x$, etc

<u>The diagram shows the graph of $f(x) = 10e^{2x}$.</u>



This graph can be seen to be neither symmetric about the $y$-axis nor resulting in the same graph if it is rotated through 180° about the origin. $f(x) = 10e^{2x}$ is **NEITHER** an odd function **NOR** an even function.

Other functions that can be seen in a similar, graphical, way to be **neither odd nor even** are

- $f(x) = x^3 + 1$, the odd function $f(x) = x^3$ shifted up by 1,
- $f(x) = \ln x$,
- $f(x) = (x - 1)^2$, the even function $f(x) = x^2$ shifted right by 1
- $f(x) = 1/(x + 2)$, the odd function $f(x) = 1/x$ shifted 2 to the left, etc

There is no need to sketch graphs, since there is an algebraic way of determining whether functions are odd, even or neither. These are outlined below.

**Algebraic Definitions**

- $f(x)$ is an EVEN function if and only if $f(x) = f(-x)$ for all $x$ in the domain of $x$.
- $f(x)$ is an ODD function if and only if $f(x) = -f(-x)$ for all $x$ in the domain of $x$.
- **else**, $f(x)$ is neither ODD nor EVEN.

The meaning of each of these definitions can be seen by using examples from above.

A look at the graph of $f(x) = x^2$ shows the symmetry with respect to the $y$-axis clearly. From this it should be noted that the function value (effectively the $y$-value) at $x$ is the same as the $y$-value at $-x$. For example, at $x = 2$, $f(x) = 4$ or, more concisely, $f(2) = 4$, and at $x = -2$, $f(-2)$ is also equal to 4. In fact, this symmetry property is such that for any value of $x$ in the domain of an even function, $f(x) = f(-x)$.

By a similar argument, for any value of $x$ in the domain of the odd function, $f(x) = x^3$, the function value ($y$-value) at $-x$ is the *negative* of the function value ($y$-value) at $+x$. For example, for $f(x) = x^3$, with $x = 2$,

$$f(-2) = (-2)^3 = -8 \text{ and } f(2) = (2)^3 = 8, \text{ so } f(2) = -(-8) = -\,f(-2)$$

This approach can be used to easily determine oddness, etc without having to sketch any graphs. Consider the following examples.

<u>1. $f(x) = \cos x$</u>
Here $f(-x) = \cos(-x) = \cos x = f(x)$, so $\cos x$ is an even function.

<u>2. $f(x) = x^2 + 2$</u>
Here $f(-x) = (-x)^2 + 2 = x^2 + 2 = f(x)$, so sin $x$ is an even function.
<u>3. $f(x) = \sin x$</u>
Here $f(-x) = \sin(-x) = -\sin x = -f(x)$, so $\sin x$ is an odd function.

<u>4. $f(x) = 1/x$</u>
Here $f(-x) = 1/(-x) = -1/x = -f(x)$, so $1/x$ is an odd function.

<u>5. $f(x) = x^3 + 1$</u>

Here $f(-x) = (-x)^3 + 1 = -x^3 + 1$. This is neither equal to $f(x)$ nor $-f(x)$, so $x^3 + 1$ is neither an odd nor an even function.

6. $f(x) = (x - 1)^2$
Here $f(-x) = (-x - 1)^2 = [(-1)(x + 1)]^2 = (x + 1)^2$. This is neither equal to $f(x)$ nor $-f(x)$, so $x^3 + 1$ is neither an odd nor an even function.

7. $f(x) = ?$
What is the only function that is **both** odd **and** even? (Answer on last page)

**Combining odd and even functions**
Even when two or more functions are combined, it is still possible to determine whether they are odd, even or neither.

Consider, for example, *odd* functions, $f(x)$ & $g(x)$ that are ***multiplied***, resulting in the function $h(x)$, i.e. $h(x) = f(x).g(x)$.

Since $f(x)$ and $g(x)$ are both odd functions $f(x) = -f(-x)$ and $g(x) = -g(-x)$, so

$$h(-x) = f(-x).g(-x) = -f(x).-g(x) = f(x).g(x) = h(x),$$

so $h(x)$, the product of two odd functions, is an even function.

This approach can be used for other combinations of odd and even functions - and for functions that are ***divided***. The results can be tabulated as follows.

| $f(x)$ | $g(x)$ | $f(x).g(x)$ or $f(x)/g(x)$. |
|---|---|---|
| EVEN | EVEN | EVEN |
| EVEN | ODD | ODD |
| ODD | EVEN | ODD |
| ODD | ODD | EVEN |

If 'NEITHER' appears anywhere in either of the first two columns then, necessarily, the product or quotient will also be 'NEITHER'.

Examples
1. $h(x) = x \sin x$

$$h(-x) = (-x).\sin(-x) = -(x).-\sin(x) = x.\sin x = h(x),$$

here the product of two odd functions results in an even function

2. $h(x) = e^x \cos x$

$$h(-x) = (e^{-x}).\cos(-x) = (e^{-x}).\cos x$$

this is neither equal to $h(x)$ nor $-h(x)$,
so the product of an even function with one that is neither odd nor even
results in a function ($h(x)$) that is neither odd nor even

Sketch graphs for both of these examples will corroborate the algebraic findings.
Now consider for example, *odd* functions, $f(x)$ & $g(x)$ that are ***added***, resulting in the function $h(x)$, i.e. $h(x) = f(x) + g(x)$.

Since $f(x)$ and $g(x)$ are both odd functions $f(x) = -f(-x)$ and $g(x) = -g(-x)$, so

$$h(-x) = f(-x) + g(-x) = - f(x) - g(x) = - (f(x) + g(x)) = - h(x),$$
so $h(x)$, the sum of two odd functions, is an odd function.

This approach can be used for the sum of other combinations of odd/even functions and for those that are ***subtracted***. The results can be tabulated as follows.

| $f(x)$ | $g(x)$ | $f(x) + g(x)$ or $f(x) - g(x)$. |
|---|---|---|
| EVEN | EVEN | EVEN |
| EVEN | ODD | NEITHER |
| ODD | EVEN | NEITHER |
| ODD | ODD | ODD |

If 'NEITHER' appears anywhere in either of the first two columns then, necessarily, the sum or the difference will also be 'NEITHER'.

Examples

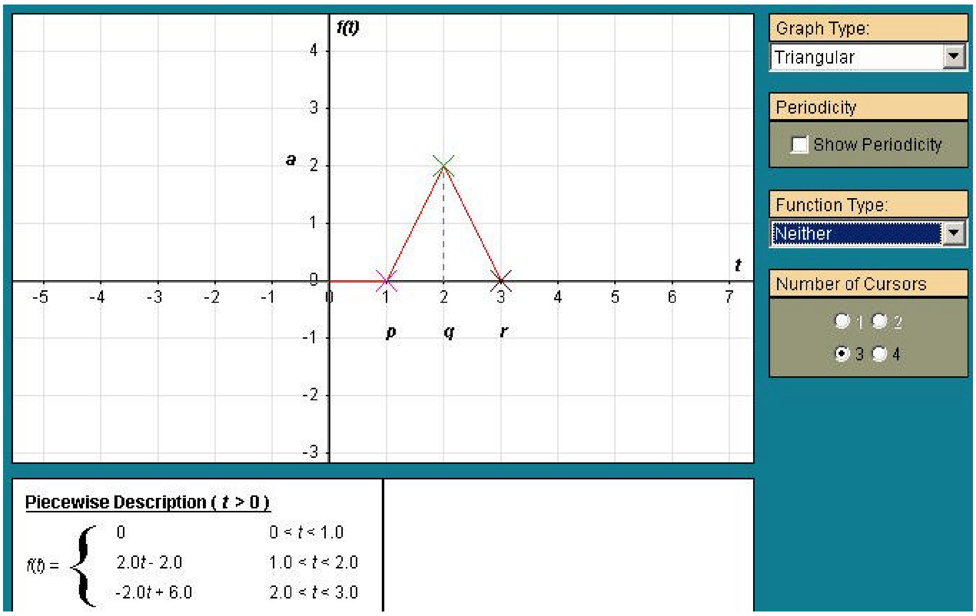
1. $h(x) = x^2 + x$

$$h(-x) = (-x)^2 + (-x) = x^2 - x$$
this is neither equal to $h(x)$ nor $- h(x)$,
so the sum of an even function and an odd function
results in a function ($h(x)$) that is neither odd nor even

. $h(x) = x^3 + x$

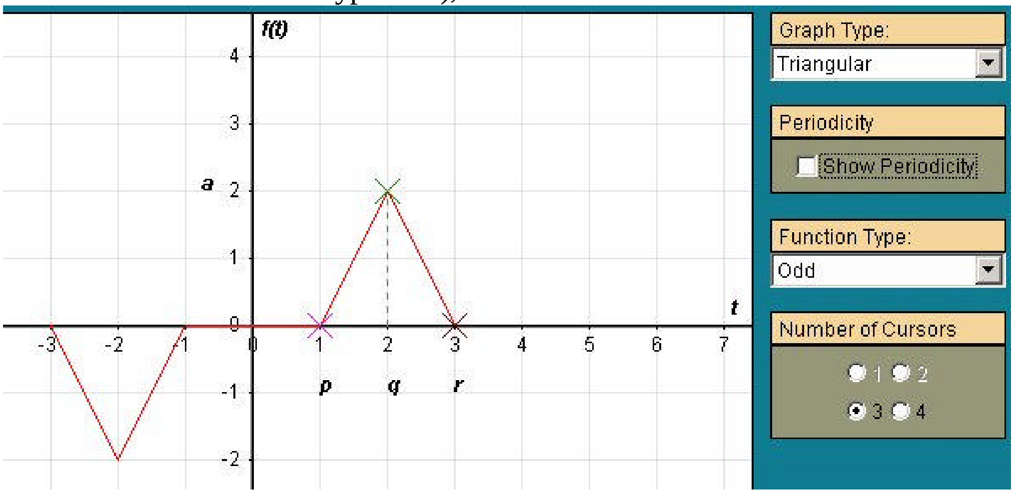
$$h(-x) = (-x)^3 + (-x) = - x^3 - x = -( x^3 + x) = - h(x)$$
so the sum of two odd functions is an odd function

Piecewise functions can also be odd, even or neither. The accompanying ***MathinSite*** applet allows the user to generate piecewise odd and even functions. Consider the following ***MathinSite*** screenshot.

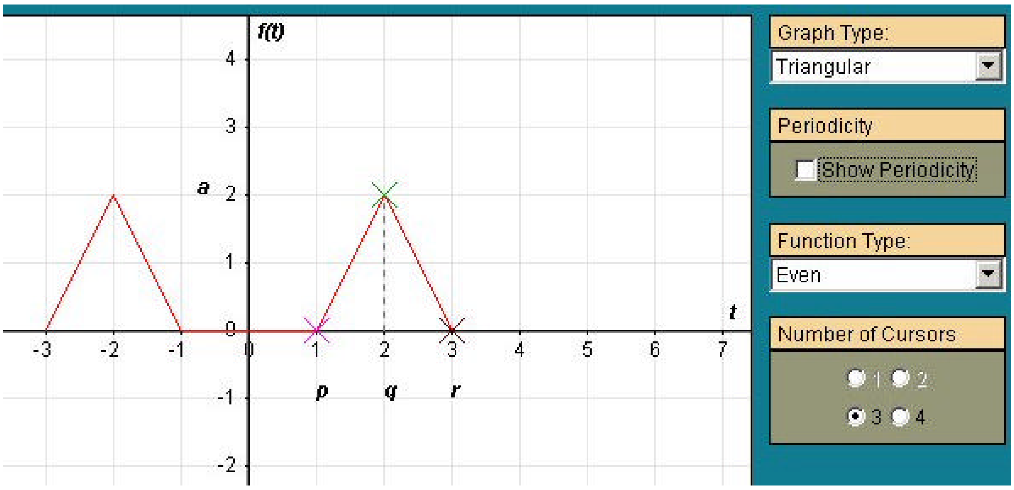



This shows a triangular waveform whose piecewise description is shown in the textbox. The function is only described from $t = 0$ to 3.

***MathinSite*** allows the user to generate odd and even waveforms from an original piecewise description. The following screenshot shows this waveform extended as an ODD function (as indicated in the 'Function Type' box),




and then the original piecewise description is extended as an EVEN function,




## Periodic Functions

A periodic function is a function that exactly and regularly repeats itself every given period, or cycle.

The definition of a periodic function mathematically is

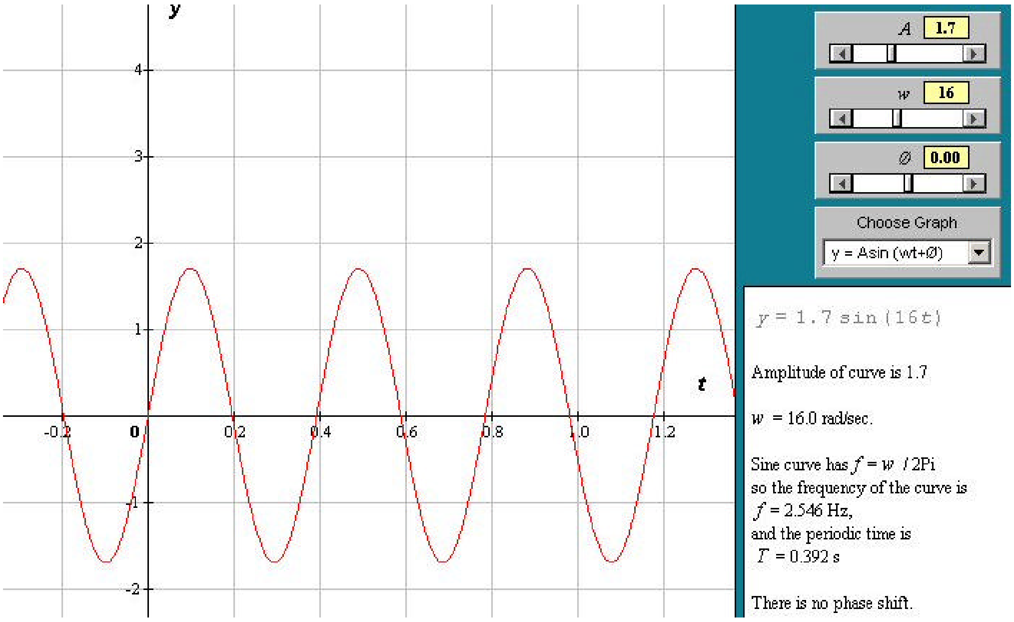
$$f(t) = f(t + T) \text{ where } T > 0 \text{ represents the period.}$$

This says that the value of the function, i.e. the '*y*' value, at $t + T$ (where $t$ could be any value) is effectively the same value as it was at $t$, so ensuring the repetitive nature of the curve for all $t$.

One of the most 'famous' periodic curves is the sine wave, a waveform that repeats every $360^o$ or $2p$ radians.

Since this theory sheet is directed particularly towards engineers and scientists, period is usually measured in terms of time - seconds for example – giving 'periodic time'.

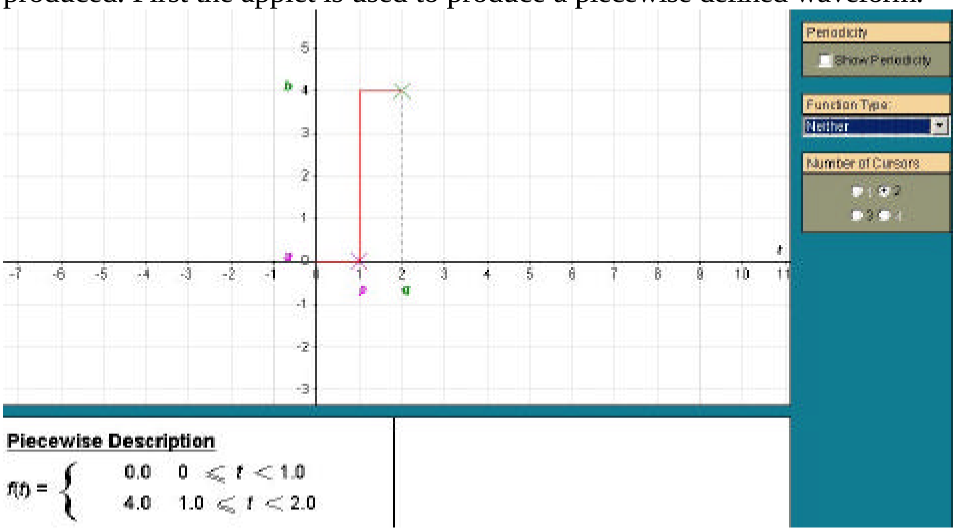
Below is a screenshot from the 'Trigonometric Functions' applet from ***MathinSite***, which shows the sinusoidal waveform $y = A\sin(wt + f)$ in which $y$ is plotted vertically and the time, $t$, is plotted horizontally.



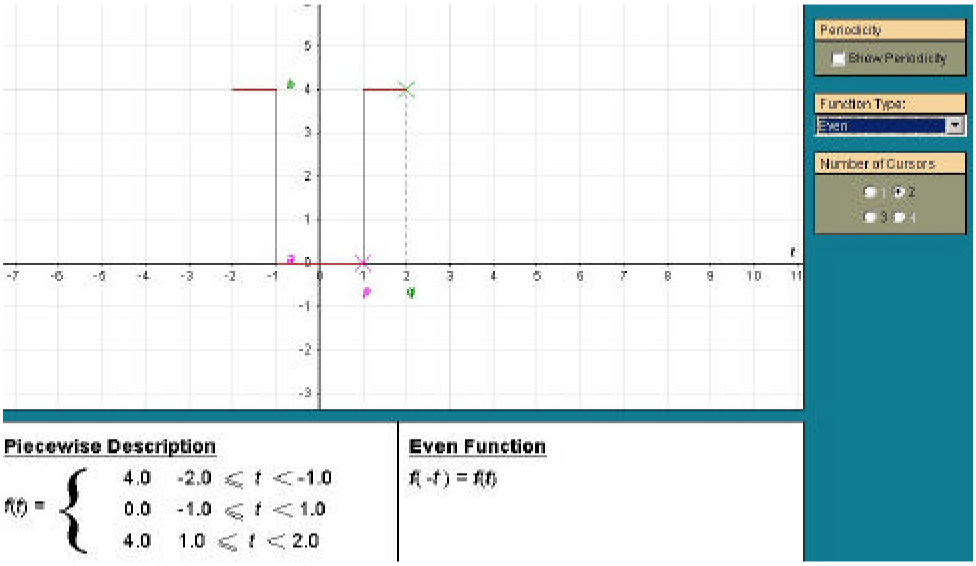

The periodic, repetitive nature of the waveform is clearly seen here. For this example, the periodic time is given as $T$ = 0.392 s (3 d.p.). The function is $f(t)$ = 1.7 sin (16$t$) and the periodicity is represented by the statement, $f(t) = f(t + 0.392)$. Choose ***any*** point on this curve; note its '$y$' value and slope and 0.392 seconds later the curve is passing through the same '$y$' value with the same slope.

One of the main aims of the accompanying applet is to investigate piecewise defined periodic functions - piecewise functions that may also be odd or even functions.

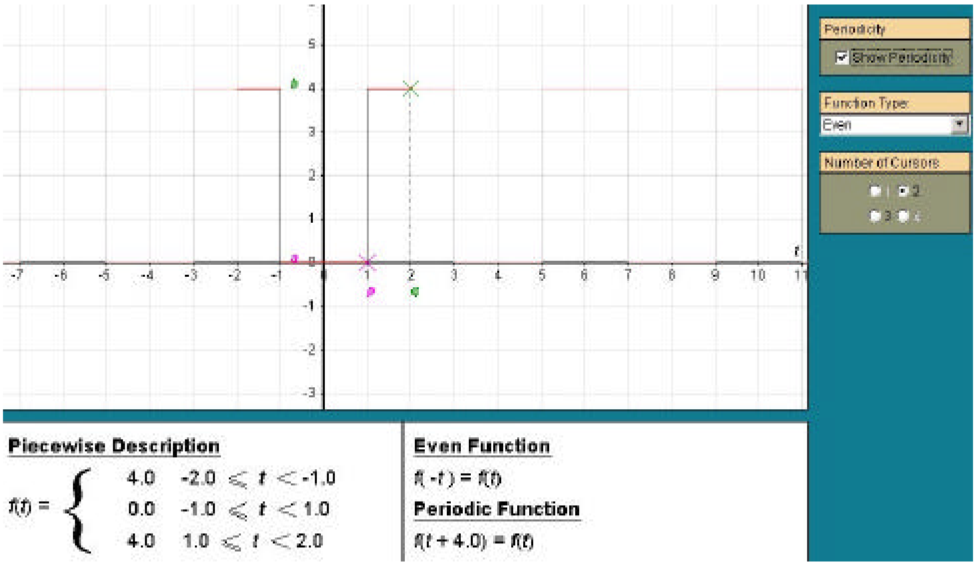
The following sequence of diagrams shows how a piecewise, periodic, even function can be produced. First the applet is used to produce a piecewise defined waveform.




This is then made an even function using the 'Function Type' drop-down box,




and finally made periodic using the 'Periodicity' checkbox,




The basic piecewise function is described from -2.0 to +2.0.
The statement $f(-t) = f(t)$ indicates that the function is an even function.
$f(t) = f(t + 4.0)$ indicates that this pulse train is periodic with periodic time 4.0 s.

The resulting periodic train of rectangular pulses is a waveform found especially in electronic engineering.

Answer to question on page 6. The only function that is both odd and even is $f(x) = 0$.