Analogue-to-Digital Pole Mapping

Learning Outcomes

After reading this theory sheet, you should

- recognise the difference between the continuous-time Laplace Transform parameter, *s*, and the *Z*-Transform discrete-time parameter, *z*
- know how to determine system poles from a transfer function
- know how the positions of system poles in the *s*-plane indicate system response and be able to identify stable or unstable systems
- know how the exact *s*→ *z* transformation, and three *s*→ *z* approximations, map pole positions from the *s*-plane to the *z*-plane.
- ascertain the effect of these transformations on system response and stability

Background knowledge required

To maximize your understanding of this theory sheet and its accompanying applet, it will help if you are familiar with first- and second-order continuous-time ***analogue*** systems and their analysis using Laplace Transforms, also discrete-time *z*-transform methods. For an investigation of the response of discrete-time ***digital*** systems, see ***MathinSite***'s 'First Order Digital Systems' applet and its accompanying theory sheet on http://mathinsite.bmth.ac.uk/html/applets.html.

Continuous-time Systems: Transfer Functions, Characteristic Equations and System Stability

Consider two different systems; first, an electrical RC series circuit and second, a mechanical mass-spring-damper system. Typical governing differential equations for these systems are, respectively,

$$\mathrm{RC}\frac{dv}{dt}+v=u \quad \text{and} \quad m\frac{d^2y}{dt^2}+R\frac{dy}{dt}+ky=u$$

where *u* is the input to each system and *v* and *y* are the outputs, respectively.

Taking Laplace Transforms of each (with zero initial conditions) and writing the transformed equations in the format

$$\frac{\textbf{L.T. of output}}{\textbf{L.T. of input}}$$

gives the ***transfer functions*** for each system as

$$\frac{\bar{v}}{\bar{u}}=\frac{1}{\mathrm{RC}s+1} \quad \text{and} \quad \frac{\bar{y}}{\bar{u}}=\frac{1}{ms^2+Rs+k}$$

Setting the denominator of the right-hand side of each to zero gives the ***characteristic equation***. This gives the position of the system's ***pole(s)*** in the complex *s*-plane.

The position of system poles in the *s*-plane is an indicator of system ***stability***. The parameter *s* is a complex number which can be plotted in an Argand Diagram (the *s*-plane). If all of a system's poles are in the left half-plane (i.e. strictly to the left of the vertical, Imaginary axis) of the Argand Diagram then the system is stable (i.e. the system response is finite – never infinite)

For the RC circuit with characteristic equation RCs + 1 = 0, there is only one pole which is situated at s = -1 / RC. This is necessarily *negative* since R and C (resistance and capacitance) are necessarily positive.

For the mass-spring-damper system, with characteristic equation $ms^2 + Rs + k = 0$, (and m, R and k all positive) there are two poles situated at

$$s = \frac{-R \pm \sqrt{R^2 - 4mk}}{2m} \text{ or } s = -\frac{R}{2m} \pm \frac{\sqrt{R^2 - 4mk}}{2m}$$

The solutions here will be of the form (depending on the relative sizes of R^2 and $4mk$)

(a) two real, distinct *negative* roots (when $R^2 - 4mk > 0$)
(b) two real, equal *negative* roots (when $R^2 - 4mk = 0$)
(c) two complex conjugate roots (when $R^2 - 4mk < 0$), in which the real part is *negative*.

A more extensive discourse on the response of second-order systems can be found on http://mathinsite.bmth.ac.uk/pdf/solsmaths.pdf

In general, systems like these, necessarily *stable* systems, will always have their poles in the left-half of the s-plane (i.e. the real part of any solution will be negative). This can be seen in the s-plane diagram in the ***MathinSite***'s **First Order Analogue to Digital Systems** applet available from http://mathinsite.bmth.ac.uk/html/applets.html in which it is possible to relate system response (in the main graphics area) to pole positions (in the small graphics area). So …

Stable continuous-time systems are those in which the system response always settles down to a steady value, usually involving transient exponential decay. In this case, ***system poles are in the left half of the s-plane***.

However, ***unstable*** continuous-time systems have their ***system poles in the right-half of the s-plane***. In unstable systems, the output becomes increasingly large, without limit, usually involving exponential growth - something which is unsustainable in real-world systems.

A system with ***poles purely on the Imaginary axis*** is a ***critically stable*** system and is one in which the system response exhibits neither exponential growth nor decay.

Since the s-plane, or ***s*-domain**, is effectively an Argand Diagram, the position of any point in the s-plane is a complex number, in this context usually written in the form $s = \sigma + j\omega$ where the real part σ is related to the exponential part of a response and is directly related to the time constant for stable systems. Necessarily, $\sigma < 0$ for stable systems. ω is related to the frequency of any oscillations that may occur in second-order systems (never in first order systems).

Mapping s-plane poles in to the z-plane

Analogue to digital (A/D) conversions are often required in computer controlled systems where, for example, an analogue signal (perhaps a voltage) has to be converted to a digital signal – the only form of signal which can be manipulated by digital computers. The *exact* transformation from the s- to the z-plane is given by

$$z = e^{sT}$$

where z, again complex, is the digital equivalent of the analogue s, and T is sampling time. Needless to say, the z-plane is also a complex plane (an Argand Diagram).

Bearing in mind that, for a continuous-time system,

$\sigma < 0$ implies a stable system (pole is in the left half plane)

$\sigma = 0$ implies critical stability and

$\sigma > 0$ implies an unstable system,

what is the equivalent set of conditions when working in the z – domain?

<u>Using the Exact transformation</u>

$$z = e^{sT},$$

so $$z = e^{(\sigma + j\omega)T} = e^{\sigma T} e^{j\omega T}$$

or $$z = e^{\sigma T}\left(\cos\omega T + j\sin\omega T\right) \quad \text{by Euler's Identity}$$

i.e. $$z = e^{\sigma T}\cos\omega T + je^{\sigma T}\sin\omega T$$

Consider now the *magnitude* of *z*, $$|z| = \sqrt{\left(e^{\sigma T}\cos\omega T\right)^2 + \left(e^{\sigma T}\sin\omega T\right)^2}$$

$$|z| = \sqrt{\left(e^{\sigma T}\right)^2\left(\cos^2\omega T + \sin^2\omega T\right)}$$

i.e. $$|z| = e^{\sigma T}$$

Now when $\sigma = 0$, $|z| = 1$, i.e. a circle, centre (0, 0) and radius 1,

and when $\sigma < 0$, $|z| < 1$, i.e. area *inside* circle, centre (0, 0) and radius 1,

and when $\sigma > 0$, $|z| > 1$, i.e. area *outside* circle, centre (0, 0) and radius 1.

So, under the s → z 'EXACT' transformation, $\sigma < 0$ (the stable region) is mapped <u>into</u> the unit circle centred on the origin in the z- plane. The stable region in the 'digital' z-plane is the inside the unit circle.

Digital systems whose poles all lie within this unit circle in the z- plane are stable systems.

<u>Using Approximate Transformations</u>

<u>1. *Using the backward transformation*</u>*,*

$$s = \frac{z-1}{zT}$$

put $z = x + jy$ to give $$s = \frac{(x-1) + jy}{(x+jy)T} = \frac{\left((x-1)+jy\right)(x-jy)}{T\left(x^2+y^2\right)}$$

Here, we are only interested in the real part of *s* (i.e. σ, since $\sigma < 0$ is the stability condition) and, for the above expression, the real part of *s* is given by

$$\text{Re}\{s\} = \sigma = \frac{x(x-1) + y^2}{T(x^2+y^2)} < 0 \quad \text{for stability}$$

Now *T*, the sampling time, and $x^2 + y^2$ are necessarily positive, so for σ to be less than zero, $$x(x-1) + y^2 < 0$$

or $$x^2 + y^2 - x < 0$$

and completing the square gives $$(x - ½)^2 + y^2 < (½)^2$$

With an equals sign, this is a circle radius ½, centre (½, 0).

With $\sigma < 0$, under the s → z 'BACKWARD' transformation, the stable region in the z – plane is <u>inside</u> the circle radius ½, centre (½, 0).

Note that since this circle is totally contained within the unit circle centred at the origin, so *a stable analogue system can never be turned into an unstable digital system using the backward transformation.*

<u>2. *Using the forward transformation*</u>*,*

$$s = \frac{z-1}{T}$$

put $z = x + jy$ to give
$$s = \frac{(x-1)+jy}{T} = \frac{(x-1)}{T} + j\frac{y}{T}$$

Here again, we are only interested in the real part of *s* (i.e. σ, since $\sigma < 0$ is the stability condition) and, for the above expression, the real part of *s* is given by

$$\mathrm{Re}\{s\} = \sigma = \frac{(x-1)}{T} < 0 \text{ for stability}$$

Now T, the sampling time, is again necessarily positive, so for σ to be less than zero,

$$x - 1 < 0$$

or
$$x < 1$$

With an equals sign, this is a vertical line at $x = 1$.

With $\sigma < 0$, under the s → z 'FORWARD' transformation, the left-half of the s – plane (the stable region) is mapped into the left-'half' plane of the z – plane to the left of x = 1.

Note that since a part of this region is *outside* the unit circle centred at the origin, *an A/D conversion, using the s → z 'forward' transformation* ***can*** *(but not necessarily) destabilise a stable system!*

<u>3. *Using the bilinear (or 'Tustin') transformation*</u>,

$$s = \frac{2}{T}.\frac{z-1}{z+1}$$

put $z = x + jy$ to give
$$s = \frac{2}{T}.\frac{(x-1)+jy}{(x+1)+jy} \times \frac{(x+1)-jy}{(x+1)-jy}$$

Once again, we are only interested in the real part of *s* (i.e. σ, since $\sigma < 0$ is the stability condition) and, for the above expression, the real part of *s* is given by

$$\mathrm{Re}\{s\} = \sigma = \frac{2}{T}.\frac{(x^2-1)+y^2}{(x+1)^2+y^2} < 0 \text{ for stability}$$

Now T, the sampling time, and $(x+1)^2 + y^2$ are necessarily positive, so for σ to be less than zero,
$$(x^2-1)+y^2 < 0$$
or
$$x^2 + y^2 - 1 < 0$$
and completing the square gives
$$x^2 + y^2 < 1$$

With an equals sign, this is a circle radius 1, centre (0, 0), i.e. an *exact* mapping on to the boundary of the stable region in the *z* – plane!

With $\sigma < 0$, under the s → z 'BILINEAR' transformation, the left-half of the s – plane (stable region) is mapped inside the unit circle, centre (0, 0) in the z – plane.

Note that since this circle is the unit circle centred at the origin, *an s → z 'bilinear' transformation can never destabilise a stable system*.

System Dynamics Considerations

The position of ***zeroes***, found by setting the denominator of the transfer function to zero, do not affect system stability and so are not covered here. In the *s*-plane, the position of poles and zeroes give an indication of system dynamics. For example, a system with poles at $s = 3 \pm 2j$ results in different time constants and periodic times of oscillation from a system with poles at $s = 2 \pm 3j$. The system response (dynamics) will be different. The same applies when mapping poles in the *z*-plane.

Under the EXACT transformation, system poles and zeroes will be mapped exactly from the *s*-plane to the *z*-plane.

Using the FORWARD transformation, it has already been shown that stable poles in the *s*-plane *may* be mapped into unstable poles in the *z*-plane, i.e. A/D conversion using the forward difference approximation can result in an unstable digital system – using the forward transformation can destabilise an otherwise stable system!

Using the BACKWARD transformation, poles cannot be placed in the *full* unit circle, so will be placed differently compared with the EXACT transformation, resulting in altered system dynamics. At least the backward transformation cannot destabilise a stable continuous-time system.

The BILINEAR mapping of the boundary of the stable region matches exactly that of the EXACT transformation; therefore the full range of dynamic response is available. However, this is no guarantee that the dynamics will be fully preserved compared with the EXACT transformation. Again though, a digitised continuous-time stable system cannot be made unstable using the BILINEAR transformation.

All of these effects can be seen using the accompanying "Analogue to Digital Pole Mapping" applet to be found on http://mathinsite.bmth.ac.uk/html/applets.html in which poles in the *s*-plane can be moved and the relevant movement of the positions of poles in the *z*-plane noted. Note that, in general, all poles found using any of the three approximation methods are placed slightly differently in the *z*-plane compared with the exact mapping. This indicates that system dynamics will be dependent on the approximation method chosen. Note also that the Tustin method generally (but not always) places its poles closest to the exact method, making it the preferred method in 'industrial' applications. Note also, when using the applet, how pole position is heavily dependent on sampling time and that too long a sampling time can destabilise systems.

An Investigation to try using hand calculation

The EXACT $s \rightarrow z$ transformation is $z = e^{sT}$ where $s = \sigma + j\omega$.

A stable analogue system has a pole at $s = -3 + 2j$. Here $\sigma = -3$ and $\omega = 2$. Determine the corresponding positions of this pole in the *z*-plane given sampling times, *T*, of 10 seconds, 1 second, 0.1 second and 0.001 second for each of the exact transformation and the three approximation methods. This is best tabulated – as in the table below. Comment on the stability of the resulting digital equivalents. (For comparison and stability purposes, it may help to determine these positions in polar form.)

Hint: Rearrange the FORWARD, BACKWARD and BILINEAR transformations, ($s = \frac{z-1}{T}$, $s = \frac{z-1}{zT}$ and $s = \frac{2}{T}.\frac{z-1}{z+1}$ respectively) making *z* the subject in each case.

For example, for the forward transformation, *z* = *s*T + 1, so when $s = -3 + 2j$, T = 10, $z = 10(-3+2j)+1 = -29+20j$. There is no need to convert this to polar form since this pole is obviously way outside the unit circle, i.e. a stable analogue system has been turned into an unstable digital system with this transformation / sampling time combination.

For $s = -3 + 2j$, z =

	Exact	Forward	Backward	Bilinear
T = 10		-29 + 20*j*		
T = 1				
T = 0.1				
T = 0.001				

Use the accompanying ***MathinSite*** applet to corroborate your answers where possible.