Name ……………………………………… Course ……………………………….
Print and use this sheet with ***MathinSite***'s 'Projectiles' applet to have a permanent record of your work.

Projectiles in a Non-Resisting Medium
The Mathematics

Aims and Objectives

- To produce a simple projectile model.
- To model the forces acting on a projectile using first-order differential equations.
- To use the 'Separation of the Variables' method to solve the differential equations.
- To produce a solution suitable to represent the trajectory of the projectile in a non-resisting medium.
- To use kinematic considerations to produce the same solutions.
- To use the accompanying software to investigate how the projectile behaves when its associated parameters are changed.
- To provide the student with some suggestions for further analysis into projectiles.

Introduction

This, the classical 'A'-level projectile model assumes, amongst others:

- the projectile is a point object (ie has no dimensions)
- there is no aerodynamic drag
- no wind
- the projectile does not spin
- gravity acts vertically downwards and has a constant value of 9.81 ms^{-2}

These assumptions lead to the parabolic trajectory usually seen in projectile problems.

As a first approximation, the above assumptions model projectiles moderately well. However, one only needs to consider examples such as

- an irregularly shaped object which is likely to tumble or spin
- a ball thrown in a cross-wind
- a table-tennis ball with backspin
- a free-flight rocket projected long-range across the face of the earth

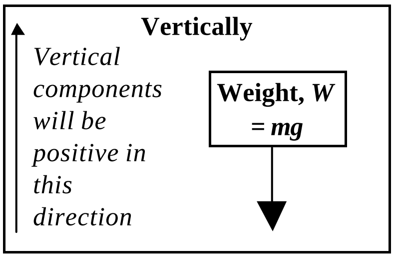
and it is easy to see that the classical parabolic projectile trajectory is unlikely to occur in reality.

Other software exists in this series, produced by the same author, that incorporates into the mathematical analysis *one* complication over and above the basic projectile model by introducing aerodynamic drag.

....but here, the basic assumptions apply.

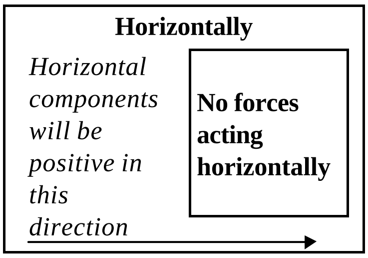
Forces acting on a projectile

With the only force considered being that due to gravity, it is possible to analyse the horizontal and vertical components of the forces acting on the body. In the subsequent analysis, the projectile is projected *upwards* and to the *right*. Consequently, the forces acting vertically are :



Here, the only force acting on the body is that due to gravity, i.e. the weight. As the body is projected upwards, the weight opposes the motion so slowing its ascent until it (instantaneously) stops at its highest point, after which the force due to gravity accelerates the projectile downwards.

The forces acting horizontally are:



Note that in this, the *classical* case, there are no forces opposing or assisting the motion horizontally. In reality aerodynamic drag, amongst other things, would slow the projectile in the horizontal direction (and the vertical direction).

Variables, Constants and Parameters

Variables

t	-	time
x	-	displacement in the horizontal direction
y	-	displacement in the vertical direction
v_h	-	horizontal component of velocity at any time t
v_v	-	vertical component of velocity at any time t

Constants

g	-	acceleration due to gravity (9.81 m s^{-2})
c	-	constant of integration

Parameters

h	-	height of projection point above 'ground level'
u	-	velocity of projection
a	-	angle of projection
b	-	angle of slope of inclined plane

Consequential Parameters

u_h	-	horizontal component of velocity of projection **(= u cos a)**
u_v	-	vertical component of velocity of projection **(= u sin a)**

The *Horizontal* Equations of Motion -Method 1

By Newton's Second Law, Force = mass x acceleration,
which, horizontally, gives

$$m\frac{dv_h}{dt} = 0 \quad \textit{(no forces horizontally)}$$

where $\frac{dv_h}{dt}$, the rate of change of horizontal velocity, is the horizontal acceleration.

Note how mass will cancel from this equation, i.e. the motion is independent of mass.

Separating the variables gives $\int dv_h = \int 0\,dt$ and integration gives $v_h = c$.

This says that the horizontal component of velocity is a constant value - it never changes - the body never slows up in the horizontal direction (not too realistic!).

So, if initially, $v_h = u_h$ then $c = u_h$ so that

$$\boxed{v_h = u_h} \text{-------------(i)} \qquad \text{for all time.}$$

Bear in mind that, since $u_h = u\cos a$, this can be written $\boxed{v_h = u\cos a}$

Since $v_h = \frac{dx}{dt}$, equation (i) becomes $\frac{dx}{dt} = u_h$, which upon direct integration of both sides with respect to t and, using the initial condition x=0 when t=0, gives

$$\boxed{x = u_h t} \text{-------------(ii)}$$

or $$\boxed{x = ut\cos a}$$

The *Vertical* Equations of Motion - Method 1

This time, Newton's 2nd Law gives

$$m\frac{dv_v}{dt} = -mg$$

Note the minus sign since the upwards direction has been chosen as positive and gravitational force acts downwards.

Dividing both sides by m and separating the variables gives $\int dv_v = -g\int dt$.

Note again how the motion of such a projectile is independent of mass.

Integration gives $$v_v = -gt + c.$$

Apply the initial condition $v_v = u_v$ when $t = 0$ giving $c = u_v$ so that

so that $$\boxed{v_v = u_v - gt} \text{-------------(iii)}$$

or, since $u_v = u\sin a$, $$\boxed{v_v = u\sin a - gt}$$

By the Chain Rule of differentiation, $\frac{dv_v}{dt} = \frac{dv_v}{dy} \times \frac{dy}{dt} = \frac{dv_v}{dy} \times v_v = v_v\frac{dv_v}{dy}$

This allows us to rework the above, but this time to find v_h in terms of x rather than in terms of t. So now

$$m\frac{dv_v}{dt} = mv_v\frac{dv_v}{dy} = -mg \text{ or, separating the variables,}$$

$$\int v_v dv_v = -g\int dy$$

or, $$\frac{v_v^2}{2} = -gy + c$$

The initial condition used here is that the projectile's speed is $v_v = u_v$ at $y = h$ (it can be projected from a point above ground level), giving

$$\frac{v_v^2}{2} = -gy + \frac{u_v^2}{2} + gh$$

therefore, multiplying throughout by 2 and rearranging gives

$$\boxed{v_v^2 = u_v^2 - 2g(y-h)} \text{---------(iv)}$$

or $$\boxed{v_v^2 = u^2 \sin^2 a - 2g(y-h)}$$

Equations (iii) and (iv) can be combined (left as an exercise) to obtain y in terms of t.

i.e. $$\boxed{y = h + u_v\, t - \frac{1}{2}gt^2} \text{------(v)}$$

or $$\boxed{y = h + u\,t \sin a - \frac{1}{2}gt^2}$$

Of more interest, especially when preparing an equation which is to be used to plot the trajectory, is the equation for y in terms of x. This can easily be found by rewriting equation (ii) as $t = \dfrac{x}{u_h}$ and substituting for t in equation (v).

So $$y = h + u_v \frac{x}{u_h} - \frac{1}{2}g\left(\frac{x}{u_h}\right)^2$$

or $$\boxed{y = h + \frac{u_v}{u_h}x - \frac{1}{2}\frac{g}{u_h^2}x^2} \text{------(vi)}$$

or $$\boxed{y = h + x\tan a - \frac{g}{2u^2\cos^2 a}x^2}$$ since $\dfrac{u_v}{u_h} = \dfrac{u \sin a}{u \cos a} = \dfrac{\sin a}{\cos a} = \tan a$

If either of these is written as $y = c + bx - ax^2$, ($a > 0$) explicitly identifying quadratic form, one can see that *all* trajectories for this model take the form of an 'inverted' parabola.

<u>Equations of Motion - Method 2</u>

The above method of the derivation of the equations of motion of such projectiles went right back to analysing the *forces* according to the assumptions made. Newton's 2nd Law, and the solution of first-order differential equations, illustrated that under these assumptions the trajectory is independent of mass (so apparently corroborating Galileo's findings when he dropped various masses from the Leaning Tower of Pisa).

A simpler method of derivation uses the basic equations of kinematics, where, automatically, any acceleration/deceleration is assumed to be constant and any consideration of mass is, necessarily, ignored.

These equations are: $v = u + at$ $\quad v^2 = u^2 + 2as$ $\quad s = ut + \frac{1}{2}at^2$

where v is the velocity at any time t, u is the initial speed and a is the acceleration.

Note that u and a are both constants, where the acceleration, a, is g, acting vertically downwards *only*. (So there is no component of a horizontally.)

For projectiles, the equations immediately above, written directly in terms of their horizontal and vertical components for a body projected upwards and to the right, are

Horizontal components:

$v_h = u_h$ $\quad v_h^2 = u_h^2$ (or, again, $v_h = u_h$) $\quad x = u_h t$

Vertical components:

$v_v = u_v - gt$ $\quad v_v^2 = u_v^2 - 2gy$ $\quad y = u_v t - \frac{1}{2}gt^2$

using the same notation as in Method 1 above.

Note that, for a body projected from a height, h, vertically above ground level, the last two equations need to have a constant value of h added to all y positions. So,

$v_v^2 = u_v^2 - 2gy$ is the same as $y = \frac{v_v^2 - u_v^2}{2g}$. So, for a body projected from height, h, all y values become $y = h + \frac{v_v^2 - u_v^2}{2g}$, or, rearranging,

$v_v^2 = u_v^2 - 2g(y - h)$ as in (iv) above, and the last equation becomes

$y = h + u_v t - \frac{1}{2}gt^2$ as in equation (v) above.

Further Analysis for you to try

Note: The maximum height of a particle projected upwards occurs when the vertical component of velocity, v_v, is zero and when the particle is at ground level, $y = 0$.

Show that:

- time to maximum height occurs when $t_{maxht} = \frac{u_v}{g} = \frac{u \sin a}{g}$
- maximum height is $y_{max} = h + \frac{u_v^2}{2g} = h + \frac{u^2 \sin^2 a}{2g}$
- the time of flight to return to horizontal ground is a solution of $\frac{1}{2}gt^2 - u_v t - h = 0$
- and hence range *on the horizontal plane* is given by $Range = \frac{u_h}{g}\left(u_v + \sqrt{u_v^2 + 2gh}\right)$

(why is the negative square root ignored?)

Trajectory over an Inclined Plane

All of the foregoing analysis is perfectly adequate for the trajectory of a projectile over a horizontal plane. Certain aspects of it are suitable also for the flight over an inclined plane - but not all. For example, the range found in (d) above is not suitable to find the range from the base of, and up, the inclined plane. In fact, the equations above are not those used in the software, since they do not cover the inclined plane case.

To determine the general case for such trajectories, the inclined plane case is analysed, with the proviso built in that if $b = 0$, then the horizontal plane situation is accommodated.

To make the analysis simpler (even so, it is still fairly complicated), instead of using the *vertical* and *horizontal* components of displacement, velocity, etc., the components *parallel* with, and *perpendicular* to, the plane are used. A sketch of the situation will show that the initial velocity components parallel and perpendicular to the plane are $u\cos(a - b)$ and $u\sin(a - b)$ respectively. Similarly, the components of gravity are $-g\sin b$ and $-g\cos b$ respectively. A geometrical analysis of the initial height shows that its vertical component is $h\cos b$ and its horizontal component is $h\sin b$. And so on.

Further Analysis for you to try

Use the basic kinematic equations and the suggestions of the previous paragraph to produce the equivalent equations for the trajectory over an inclined plane. In particular, show that the time up to the moment the projectile impacts upon the plane is

- $t' = \dfrac{u\sin(a-b) + \sqrt{u^2\sin^2(a-b) + 2gh\cos^2 b}}{g\cos b}$,

and the range from the base of, and up, the inclined plane is given by

- $Range = h\sin b + ut'\cos(a-b) - \dfrac{1}{2}gt'^2\sin b$

See if you can develop the equation for y in terms of x (equivalent to equation (vi)). For the inclined plane case, this was the equation used to produce the plots in the accompanying software.