Projectiles in a Resisting Medium
The Mathematics

Aims and Objectives

- To produce a projectile model incorporating a simple approximation for aerodynamic drag.
- To model the forces acting on a projectile using first-order differential equations.
- To use the 'Separation of the Variables' method to solve the differential equations.
- To produce a solution representing a more realistic trajectory of a projectile in a resisting medium than the classical 'no-drag' case.
- To use Maclaurin's Theorem to reduce the solution to the simpler case of projectiles in a non-resisting medium.
- To use the accompanying software and tutorial sheet to investigate how the projectile behaves when its associated parameters are changed.
- To provide the student with some suggestions for further analysis into projectiles.

Introduction

The classical 'A'-level projectile modelling assumptions include:

- the projectile is a point object (ie has no dimensions)
- there is no aerodynamic drag
- no wind
- the projectile does not spin
- gravity acts vertically downwards and has a constant value of 9.81 ms^{-2}

These lead to the parabolic trajectory seen in this type of projectile problem.

As a first approximation, the above assumptions model projectiles moderately well. However, consider examples such as
- a large irregularly-shaped, large object which is likely to tumble or spin
- a ball thrown in a cross-wind
- a table-tennis ball with backspin
- a free-flight rocket projected long-range across the face of the earth

and it is easy to see that the classical parabolic projectile trajectory is unlikely to occur in reality.

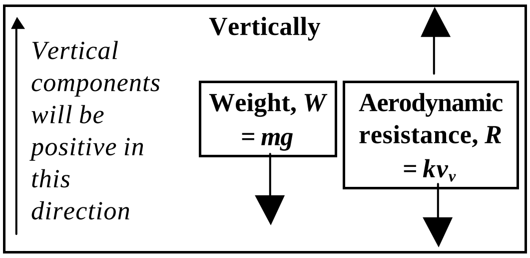
This paper supplements the 'A'-level mathematical analysis by incorporating aerodynamic drag into the basic projectile model. The above assumptions involving gravity, no spin or wind and the object being a point mass will remain.

Forces acting on a projectile

The assumption of no aerodynamic drag is replaced here, at least initially, by:
- ***Aerodynamic drag is assumed to be proportional to the velocity of the projectile***.

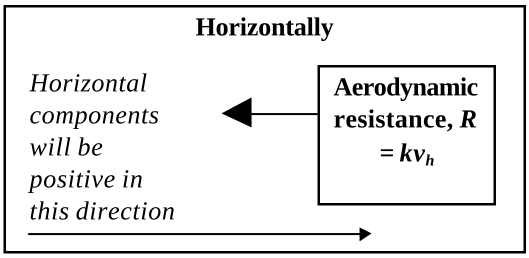
This is the simplest assumption in such cases and can be shown experimentally to be a 'reasonable' approximation. (Have you noticed at the seaside, how much more difficult it is to *run* into the sea, rather than walk – i.e. the retarding force is greater at faster speeds.) A more complicated (but even more realistic) model uses the retarding force as being proportional to the velocity *squared* (however, this case does not have an analytical solution and has to resort to numerical methods for its solution).

With only the forces due to gravity and aerodynamic drag being considered, it is possible to analyse the horizontal and vertical components of these forces acting on the body. In the subsequent analysis, the projectile is projected *upwards* and to the *right*. Consequently, the forces acting vertically are:



Here, k is the constant of proportionality and is related, amongst other things, to the cross-sectional area of the projectile and to the viscosity of the fluid (including that of air) through which the projectile passes. R will act downwards if the motion is upwards and upwards if the motion is downwards.

The only force acting horizontally is:



Since sideways motion is always to the right, R will always act to the left, opposing the motion. Note that in the *classical* case, there are no forces acting horizontally.

Variables, Constants and Parameters

Variables

t	-	time
x	-	displacement in the horizontal direction
y	-	displacement in the vertical direction
v_h	-	horizontal component of velocity at any time t
v_v	-	vertical component of velocity at any time t

Constants

g	-	acceleration due to gravity (9.81 m s^{-2})
c	-	constant of integration

Parameters

m	-	mass of the projectile
k	-	constant of proportionality (related to viscosity of fluid, size of object)
h	-	height of projection point above 'ground level'
u	-	velocity of projection
a	-	angle of projection

Consequential Parameters

u_h	-	horizontal component of velocity of projection (= $u \cos a$)
u_v	-	vertical component of velocity of projection (= $u \sin a$)

The *Horizontal* Equations of Motion

Newton's Second Law (namely, Force = mass x acceleration) gives, horizontally,

$$m\frac{dv_h}{dt} = -kv_h$$

where $\frac{dv_h}{dt}$, the rate of change of horizontal velocity, is the horizontal acceleration.

Note the minus sign, since the motion is to the right and the aerodynamic drag *opposes* the motion.

Separating the variables gives $\int \frac{dv_h}{v_h} = -\frac{k}{m}\int dt$,

Integration gives $\ln v_h = -\frac{kt}{m} + c$.

Apply the initial condition $v_h = u_h$ when $t = 0$ giving $\ln u_h = c$ so that

$$\ln v_h = -\frac{kt}{m} + \ln u_h$$

or $\ln v_h - \ln u_h = -\frac{kt}{m}$

so $\ln\frac{v_h}{u_h} = -\frac{kt}{m}$

Exponentiating both sides and then multiplying both sides by u_h gives

$$\boxed{v_h = u_h\, e^{-\frac{kt}{m}}} \text{--------------(i)}$$

By the Chain Rule, $\frac{dv_h}{dt} = \frac{dv_h}{dx}\times\frac{dx}{dt}$. But $\frac{dx}{dt} = v_h$, so $\frac{dv_h}{dt} = \frac{dv_h}{dx}\times v_h = v_h\frac{dv_h}{dx}$

This allows us to rework the above, but this time to find v_h in terms of x rather than t. So now

$$m\frac{dv_h}{dt} = mv_h\frac{dv_h}{dx} = -kv_h$$

leading to $m\frac{dv_h}{dx} = -k$ or, separating the variables,

$$\int dv_h = -\frac{k}{m}\int dt$$

so that $v_h = -\frac{k}{m}x + c$

The initial condition used here is that the projectile's velocity is $v_h = u_h$ when $x = 0$, giving $u_h = c$,

therefore $\boxed{v_h = u_h - \frac{k}{m}x}$ ---------(ii)

Equations (i) and (ii) can be combined (left as an exercise) to obtain x in terms of t.

i.e. $$x = \frac{u_h\, m}{k}\left(1 - e^{-\frac{kt}{m}}\right) \text{------(iii)}$$

These three results, (i), (ii) and (iii) can already provide some important information on the motion of the projectile. But first, a digression on the time constant.

The Time Constant, *t*

Any exponentially decaying term can be written using $e^{-\frac{t}{t}}$, where t (the Greek letter 'tau') is called the ***time constant***. Consider the following table:

t	$e^{-\frac{t}{t}} \times 100\%$
0	e^0 x 100 =100
t	e^{-1} x 100 = 36.7879
$2t$	e^{-2} x 100 = 13.5335
$3t$	e^{-3} x 100 = 4.9787
$4t$	e^{-4} x 100 = 1.8316
$5t$	e^{-5} x 100 = 0.6738

$5t$ is an important value!

The right hand column shows that the value of $e^{-\frac{t}{t}}$ varies from 100% at time zero down to about 0.5% by $5t$.

The implication is that by $5t$, the contribution of $e^{-\frac{t}{t}}$ has died away to 'practically nothing'.

The exponentials in (i) and (iii) are both of the form $e^{-\frac{kt}{m}}$ so for our problem, $t = \frac{m}{k}$.

Equation (i) indicates therefore that, by $5m/k$, the horizontal component of velocity is practically zero and equation (iii) shows that, by this time, the horizontal displacement has effectively attained a maximum (limiting) value of $x = \frac{u_h\, m}{k}$. This means that the projectile has no forward velocity and will just *fall* to ground level.

These values are attained only if the projectile has ***not*** *hit the ground before 5m/k.*

The *Vertical* Equations of Motion

This time, Newton's 2nd Law gives

$$m\frac{dv_v}{dt} = -(mg + kv_v)$$

Note the minus sign, since the body is projected upwards and the aerodynamic drag and the weight *oppose* the motion.

Separating the variables gives $\int \frac{dv_v}{mg + kv_v} = -\frac{1}{m}\int dt$, or

$$\int \frac{kdv_v}{mg + kv_v} = -\frac{k}{m}\int dt$$

Integration gives $$\ln(mg + kv_v) = -\frac{kt}{m} + c\,.$$

Apply the initial condition $v_v = u_v$ when $t = 0$ giving $\ln(mg + ku_v) = c$ so that

$$\ln(mg + kv_v) = -\frac{kt}{m} + \ln(mg + ku_v)$$

or $$\ln(mg + kv_v) - \ln(mg + ku_v) = -\frac{kt}{m}$$

so $$\ln\left(\frac{mg + kv_v}{mg + ku_v}\right) = -\frac{kt}{m}$$

Exponentiating both sides and then multiplying both sides by $(mg + ku_v)$ gives

$$mg + kv_v = (mg + ku_v)e^{-\frac{kt}{m}}$$

so that $$\boxed{v_v = \frac{1}{k}\left\{(mg + ku_v)e^{-\frac{kt}{m}} - mg\right\}} \text{-------------(iv)}$$

Again, by the Chain Rule, $\frac{dv_v}{dt} = \frac{dv_v}{dy} \times v_v = v_v \frac{dv_v}{dy}$

This allows us to rework the above, but this time to find v_h in terms of x rather than in terms of t. So now

$$m\frac{dv_v}{dt} = mv_v \frac{dv_v}{dy} = -(mg + kv_v) \text{ or, separating the variables,}$$

$$\int \frac{mv_v}{mg + kv_v} dv_v = -\int dy \qquad \text{(A)}$$

$$\frac{m}{k}\int \frac{kv_v}{mg + kv_v} dv_v = -\int dy$$

$$\frac{m}{k}\int \frac{(mg + kv_v) - mg}{mg + kv_v} dv_v = -\int dy$$

$$\frac{m}{k}\int 1 - \frac{mg}{mg + kv_v} dv_v = -\int dy$$

$$\frac{m}{k}\int dv_v - \frac{m^2 g}{k^2}\int \frac{k}{mg + kv_v} dv_v = -\int dy$$

(*This could also have been obtained using algebraic long-division on the integrand in equation* (A) *above.*)

so that $$\frac{m}{k}v_v - \frac{m^2 g}{k^2}\ln(mg + kv_v) = -y + c$$

The initial condition used here is that the projectile's speed is $v_v = u_v$ at $y = h$ (it can be projected from a point above ground level), giving

$$\frac{m}{k}u_v - \frac{m^2 g}{k^2}\ln(mg + ku_v) = c$$

Therefore, substituting for c and rearranging gives

$$\boxed{y = h + \frac{m^2 g}{k^2}\ln\left(\frac{mg + kv_v}{mg + ku_v}\right) - \frac{m}{k}(v_v - u_v)} \text{---------(v)}$$

Equations (iv) and (v) can be combined (left as an exercise) to obtain y in terms of t.

i.e. $$\boxed{y = h + \frac{m}{k}\left\{(u_v - gt) + \frac{mg}{k} - \left(\frac{mg + ku_v}{k}\right)e^{-\frac{kt}{m}}\right\}} \text{------(vi)}$$

Of more interest, especially when preparing an equation that is to be used to plot the trajectory, is the equation for y in terms of x.

Equation (iii) can be rewritten as $e^{-\frac{kt}{m}} = 1 - \frac{kx}{mu_h}$, so that equation (iv) becomes

$v_v = \frac{1}{k}\left\{(mg + ku_v)\left(1 - \frac{kx}{mu_h}\right) - mg\right\}$ which can be substituted into (v) to give

$$y = h + \frac{m^2 g}{k^2}\ln\left(\frac{mg + k\frac{1}{k}\left\{(mg + ku_v)\left(1 - \frac{kx}{mu_h}\right) - mg\right\}}{mg + ku_v}\right) - \frac{m}{k}\left(\frac{1}{k}\left\{(mg + ku_v)\left(1 - \frac{kx}{mu_h}\right) - mg\right\} - u_v\right)$$

Fortunately, this simplifies to become

$$\boxed{y = h + \frac{m^2 g}{k^2}\ln\left(1 - \frac{kx}{mu_h}\right) + \frac{x}{ku_h}(mg + ku_v)} \text{--------(vii)}$$

Equation (vii) is the equation used to produce the plots in the accompanying software.

<u>What happens when *k* is small?</u>

Since k is directly related, amongst other things, to the viscosity of the fluid through which the projectile passes, the larger the value of k the more 'sticky' the fluid. Conversely, in the limit, as $k \to 0$, the less sticky the fluid and the classical case of projectiles will result.

Unfortunately, it is not possible to substitute $k = 0$ into some of the above equations since, for example, division by zero results and a solution is therefore not defined.

In such cases, it is necessary to use the Maclaurin expansions of the exponential and logarithmic functions and investigate their values as $k \to 0$.

The following Maclaurin expansions are needed:

$$e^{-\frac{kt}{m}} = 1 - \frac{kt}{m} + \frac{k^2 t^2}{2m^2} - \ldots \quad \text{and} \quad \ln\left(1 + \frac{kv}{mg}\right) = \frac{kv}{mg} - \frac{k^2 v^2}{2m^2 g^2} + \ldots$$

It is left as an exercise to show that the above equations become

(i)'	$v_h = u_h$	(ii)'	$v_h = u_h$
(iii)'	$x = u_h t$	(iv)'	$v_v = u_v - gt$
(v)'	$v_v^2 = u_v^2 - 2g(y - h)$	(vi)'	$y = h + u_v t - \tfrac{1}{2}gt^2$

(vii)' $y = h + \frac{u_v}{u_h}x - \frac{g}{2u_h^2}x^2$ or, in terms of u and a, $y = h + x\tan a - \frac{gx^2 \sec^2 a}{2u^2}$

It is interesting to note that the mass of the object *also* disappears from the equations when k (viscosity) is taken out. Perhaps Galileo was only partly right?!

Further Analysis

- The maximum height of a particle projected upwards occurs when the vertical component of velocity, v_v, is zero. Show that the

  (a) time of flight to any horizontal displacement x is $t = -\frac{m}{k}\ln\left(1 - \frac{kx}{mu_h}\right)$

  (b) time to maximum height occurs when $t_{\max} = \frac{m}{k}\ln\left(1 + \frac{ku_v}{mg}\right)$

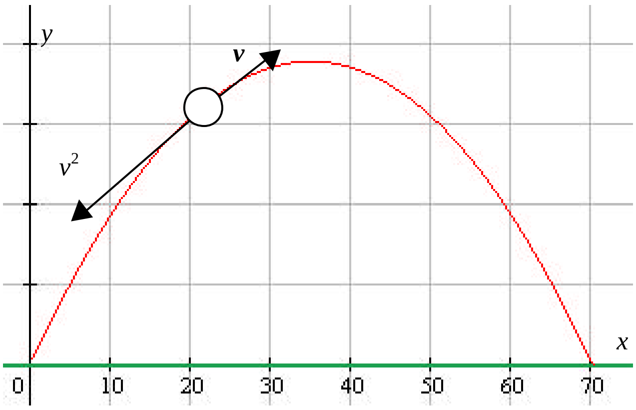
  (c) maximum height is $y_{\max} = h + \frac{m}{k}\left(u_v - gt_{\max}\right)$

- How do you suppose that the software uses equation (vi) to find the time of flight (i.e. the value of t when $y = 0$), and hence the horizontal range - information both given in the lower box on-screen? (No, it doesn't use the Newton-Raphson method.)

- How are the differential equations affected if a wind blows directly from behind the point of projection? Assume that the wind exerts a *constant* force of magnitude W in the positive x direction only. Try solving these equations in the manner used in this paper. What if it is a head-wind?

- Sometimes the resistance to motion is modelled as being proportional to the velocity *squared*. This case is altogether more complicated than the one shown above and an introduction to its solution follows on page 8.

The Case when Aerodynamic Drag is proportional to Velocity *squared*



The diagram shows a projectile travelling from left to right. The aerodynamic drag is proportional to the ***magnitude*** v^2 and opposes the motion i.e. it is in a direction of $-\boldsymbol{v}$, where $\boldsymbol{v}$ is the velocity vector of the projectile at any time t.

So that aerodynamic drag $= -kv^2\hat{\mathbf{v}}$ where

- v^2 is given by $v^2 = v_h^{\ 2} + v_v^{\ 2}$
- k is the constant of proportionality and is related again, amongst other things, to the viscosity of the medium through which the projectile is travelling.
- $\hat{\mathbf{v}}$ is a *unit* vector in the direction of $\boldsymbol{v}$ (- $\hat{\mathbf{v}}$ just gives a *direction* to the *magnitude* v^2)

Since $\hat{\mathbf{v}}$ is a unit vector in the direction of $\boldsymbol{v}$, it is equal to the vector $\boldsymbol{v}$ divided by the magnitude of vector $\boldsymbol{v}$ (i.e. vector $\boldsymbol{v}$ but with its magnitude, $|\boldsymbol{v}|$, 'divided out').

Let $\hat{\boldsymbol{i}}$ be a unit vector in the x-direction and $\hat{\boldsymbol{j}}$ be a unit vector in the y-direction.

The ***vector*** equation of motion (by Newton's 2nd Law) will now be

$$m\frac{d\mathbf{v}}{dt} = -kv^2\hat{\mathbf{v}} - mg\,\hat{\boldsymbol{j}} = -kv^2\frac{\mathbf{v}}{|\mathbf{v}|} - mg\,\hat{\boldsymbol{j}}$$

In component form, this will be

$$m\frac{d\left(v_h\hat{\boldsymbol{i}} + v_v\hat{\boldsymbol{j}}\right)}{dt} = -k\left(v_h^{\ 2} + v_v^{\ 2}\right)\frac{\left(v_h\hat{\boldsymbol{i}} + v_v\hat{\boldsymbol{j}}\right)}{\sqrt{v_h^{\ 2} + v_v^{\ 2}}} - mg\,\hat{\boldsymbol{j}}$$

and separating this out into individual components results in

$$m\frac{dv_h}{dt} = -kv_h\sqrt{v_h^{\ 2} + v_v^{\ 2}} \qquad \text{horizontally, and}$$

$$m\frac{dv_v}{dt} = -kv_v\sqrt{v_h^{\ 2} + v_v^{\ 2}} - mg \qquad \text{vertically}$$

… two simultaneous, non-linear first-order differential equations!

These equations do not have an analytical solution i.e. you can't integrate, etc. as in the previous case (pages 1 to 7) to obtain a 'nice' formula answer. The equations have to be solved using a numerical method (4th order Runge-Kutta, for example).

The analysis required here is beyond the scope of what was intended in this document. A separate paper may appear sometime in the future, but it may appear sooner if a volunteer has the time to contribute such a paper to this site (***with*** the accompanying applet and tutorial sheet!).