Second Order Linear Systems
Tutorial Worksheet

Learning Outcomes
- To develop, through experimentation, an understanding how a second-order linear system with constant input responds from changing the position of its system poles and the magnitude of its constant input.
- To enable "what if ... ?" questions about the second-order systems to be answered.

Introduction
Second-order linear systems have important applications in the study of topics in, amongst others, mechanical and electronic engineering. In particular, they occur in the analysis of mechanical vibrating systems and electrical circuits (for the mathematical modelling of these systems, see the accompanying "Second-order linear systems - Theory Sheet").

This applet allows investigation of a second-order linear system with a constant forcing term as input (driving force) to the system and zero initial conditions. The response of the system is investigated by dragging 'crosses' (the system's poles) around a graphics area with the mouse and by using a Windows-type slider bar to change the magnitude of the driving force.
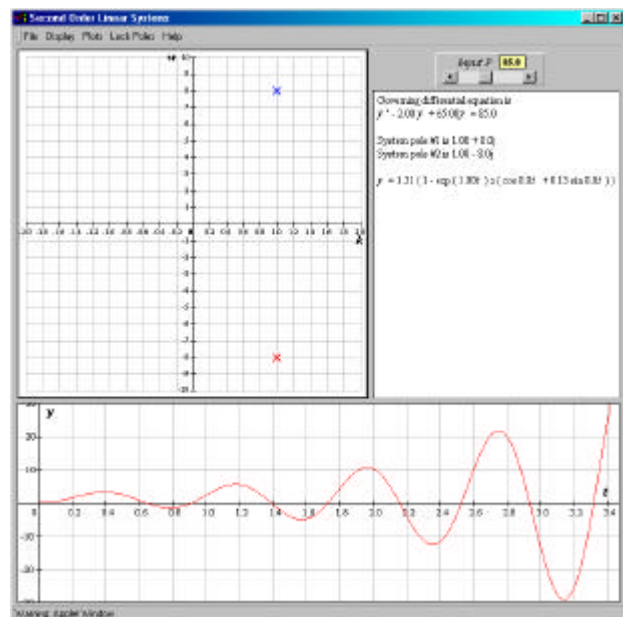
To fully appreciate the working of this applet you should be familiar with the theory of systems and poles. The applet is not intended to teach you the subject but to give you a feel for how pole positions affect system response. For the background theory however see the accompanying theory sheet, "Second Order Linear Systems – The Mathematics".

Loading the Software
Select and run "Second Order Linear Systems" from the choice of applets offered by the web site from which this document was downloaded.

Running the applet
When loaded, you see the applet window as shown. The default values are $P = 85$ with the system poles placed at $1 + 8j$ and $1 - 8j$. The corresponding system response is displayed in the graph at the bottom of the screen (system output on the vertical axis, time on the horizontal axis) with the analytical solution in the text area on the right.

The slider bar on the upper right of the applet window allows the user to change the value of parameter P (related to the magnitude of the constant input to the system). Clicking the arrow boxes at either end of the slider bar effect *small changes* in the parameter value. Clicking in the main part of the box of the slider bar makes *larger changes*. Clicking, holding and moving the slider in the slider bar can make any sort of change.

For the associated 'Second Order Systems' applet, visit http://mathinsite.bmth.ac.uk/html/applets.html

Changing the value of P by moving the slider and moving one or the other of the poles by dragging with the mouse is the only means of user-input.

The software gives you the option of showing single plots or multiple plots (under 'Plots' in the pull down menu structure at the top of the applet window). It also allows you to 'drag' the plot around (to see areas not covered by the original window on the plot) by holding down the mouse button whilst at the same time moving the mouse over the plot. The plot can be recentred at any time using the 'Centre on Origin' option, also under 'Plots'.

The Tutorial

Changing *k only*

- Drag either one of the poles (red, or blue) onto the real (horizontal) axis of the *s*-plane – the other pole should move as a mirror image on the other side of the horizontal axis (remember poles off the real axis appear as complex conjugates). When both poles come together on the horizontal axis, a larger, magenta-coloured, cross will appear. Now from the "Lock Poles" pull-down menu, choose "Vertically". When you move either pole now it will remain on the horizontal axis. Change the value of *P* to 100 and move both poles along the axis until one is at +0.4 and the other is at +0.6. This will require a steady hand!

- Drag the plot as necessary in the lower graphics window to follow the red curve away from the origin. You should clearly see the output in this window demonstrating exponential growth. Keep a record of this by sketching the resulting response curve, the pole positions and the analytical solution on paper. Label this as *Figure 1.* so that you can cross-reference it to these notes.

- Increasing either value of *k* (by moving either or both poles further to the right) will result in a 'faster' rate of growth. Note that the analytical solution of the differential equation in the text box shows two exponentially increasing terms. What is the relationship between the pole positions and the analytical solution?
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- Move one of the poles just to the left of the vertical axis. You may have to drag the lower window to view the response curve (or press "Centre on Origin" under "Display" in the menu system). You should still see exponential growth in the lower window *since one of the poles is still to the right of the origin* (i.e. *k* is still greater than zero).

- Move the other pole to the left of the vertical axis also. The solution to the governing differential equation – in the text box on the right – now shows two exponential decay terms. With a constant input this will result in exponential growth to a limiting value. How far the poles are placed along the horizontal axis from the origin governs how 'quickly' the solution curve takes to reach this limiting value. Too close to the origin and the growth to a limit will take too 'long' to be seen in the range allowed. Moving the poles further to the left and/or changing the value of *P* will bring the steady state value into range. Position the poles so that you get a response showing the system settling down to the steady state value within a screen's width (poles at –2.0 and –1.8 with *P* = 100 should achieve this). Keep a record of this by sketching the resulting response curve, the pole positions and the analytical solution on paper. Label this as *Figure 2.* so that you can cross-reference it to these notes.

- You should now be aware that if *any* single system pole lies to the right of the vertical axis then exponential growth will result and the system is said to be *unstable*. If both (or all, in higher order systems) of the system poles lie to the left of the vertical axis then only exponential decay will occur and the system is said to be *stable*.

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Print and use these sheets with *MathinSite*'s 'Second Order Systems' applet to have a permanent record of your work.

- You may like to sketch the result from another set of pole positions in order to illustrate further what you have learned from the above. Label this ***Figure 2a***.

The effect of changing k on the time constant, t

- The exponentials in the solutions you have just seen on-screen are both in the form of exponential *decay* to a limiting value. From the pole positions, it is possible to determine the *time constants* for each decay term (simply from $1/k$ for each pole). Time constants are discussed in the accompanying Theory Sheet. You should find by dragging the curve in the lower window, that the steady state (constant) output is achieved in about five times the larger time constant. Try this. Move both poles on the horizontal axis down to –2.0. Follow the response curve in the lower graphics area to see that the curve levels out by about $t = 2.5$ (5 times 0.5). Notice that moving the slider to change the magnitude of the constant input does not change the time taken to reach steady state. Try it.

- Sketch this situation showing the position of the poles, the analytical solution and the system output, marking on the horizontal axis the value of 5 times the larger time constant i.e. time to reach steady state along the k axis. Label this ***Figure 3***.

- Moving both poles closer to the origin will increase the exponentials' time constants. Watch how the time to steady state increases as you move both poles towards the origin. Since the software does not permit unlimited scrolling to the right, eventually you will be unable to see the steady state. [*Note that the two poles each have their own time constant. The one closer to the origin necessarily has the larger time constant and so will dictate the time for the system to eventually settle down to its steady state value*.]

- You can now release the vertical lock on the poles by choosing "Off" from the "Lock Poles" pull-down menu.

The effect of changing w only

- If you still have both of the poles on the horizontal axis, drag one on top of the other to obtain the larger, magenta-coloured cross again. If you have switched the "Lock Poles" to "Off" already you should now be able to pull the poles away from the horizontal, k, axis (as a complex conjugate pair).

- Drag either of the poles to $-0.5 + 5j$ (this may need a steady hand!), the other will automatically fix on the complex conjugate $-0.5 – 5j$. Choose "Horizontally" from the "Lock Poles" menu to ensure that the k value remains constant, allowing you to move the poles in the w direction only.

- The exponentials in the solutions you have just seen on-screen are both in the form of exponential decay with a single time constant $1/0.5 = 2.0$. So the steady state should occur within about 10 seconds (five times the time constant).

- The value of w here is 5. So the periodic time, T, $(2\pi/w)$ is about 1.25 seconds. Notice that moving the slider to change the magnitude of the constant input does not change the frequency of the oscillations. Since the time constant is $1/0.5 = 2$, the time for the oscillations to decay is about 10 seconds. You should therefore see about 8 complete oscillations (with successively smaller amplitude) before steady state is reached. Keep a record of this by sketching the resulting response curve, the pole positions and the analytical solution on paper. Label this ***Figure 4.*** so that you can cross-reference it to these notes.

- Moving both poles closer to the horizontal axis reduces the value of w, hence the frequency, f. As the poles reach the horizontal axis where $w = 0$, the oscillations will eventually disappear. Why does this happen?

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- This is the case you have already seen when you locked the poles on the horizontal axis. Note how the time to steady state doesn't change while you are changing the position of the poles vertically (only).

Poles on the Vertical Axis

- In the last section you started by placing one of the poles at –0.5 + 0.5*j* and locked the poles horizontally. Place one of the poles at –0.5 + 0.5*j* again, but lock the poles vertically (from the "Lock Poles" pull-down menu). Move either one of the poles until they are both on the vertical axis. At this point the poles have a *k* value of zero.

What effect does this have on the "exponential part" of the analytical solution?

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And what type of system output results?

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- Keep a record of this result by sketching the response curve, the pole positions and the analytical solution on paper. Label this *Figure 5.* so that you can cross-reference it to these notes.

- With the poles on the vertical axis at +0.5j and –0.5j, lock the poles horizontally and move them slowly *down* the vertical axis.

What is the effect on the system output when you do this?

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Remember to turn "Off" locked poles from the "Lock Poles" menu before you experiment further with this applet.

Summary

In a second-order system,

- poles always appear either (a) as complex conjugate pairs or (b) both on the real axis
- if *any* pole lies to the right of the vertical axis (i.e. in the right half-plane), then the system will necessarily contain at least one exponential growth term and is said to be an *unstable* system.
- if *both* (or *all*, in the case of higher order systems) poles lie in the left half-plane, then the system will necessarily contain exponential decay terms only (with no exponential growth terms) and is said to be a *stable* system.
- if the poles appear as a complex conjugate pair on the vertical axis (but not both at the origin), there will be neither exponential growth nor decay and the system responds with only pure oscillations (simple harmonic motion, SHM). This system is said to be critically stable.
- if both poles are at the origin, there are neither exponential nor sinusoidal terms; the analytical solution (hence output) for such a system is a pure quadratic increase ($P > 0$).
- poles vertically further away from the horizontal axis will relate to higher frequency oscillations than poles closer to the horizontal axis. Poles *on* the horizontal axis have zero frequency i.e. do not contribute any oscillatory effect to the system output.
- moving the poles horizontally away from the vertical axis results in more rapid exponential growth (or decay, depending whether the pole is in the right or left half-plane, respectively). Conversely moving the poles horizontally towards the vertical axis relates to slower exponential growth (or decay). Poles on the vertical axis relate to neither growth nor decay.

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Exercises

1. The system represented in this applet uses zero initial conditions in *all* cases, i.e. $y = 0$ and $dy/dx = 0$. How does this relate to the way in which your system output graphs are drawn round about $t = 0$?

2. What is the time to steady state for the systems with poles at
   - -0.4 + 2*j* and –0.4 –2*j*
   - -1.0 and –0.5
   - 0.4 + 2*j* and 0.4 –2*j*

   Beware trick questions!

3. Determine the times to steady state and the periodic times for the systems with poles at
   - -0.2 + j and –0.2 – j
   - -1 + 10*j* and –1 –10*j*
   - -10 + *j* and –10 – *j*

   In each case determine how many oscillations will occur before steady state (see the accompanying theory sheet). Sketch the curves. Use the applet where possible to corroborate your answer.

4. Determine the time constants for systems with the following poles:
   - -1 and –1.2
   - -1.5 (repeated)
   - +1 and –1.2

   How long is the transient time in each case?
   Beware trick questions!

5. Write down the analytical solutions for systems with the following poles (use the applet for each case):
   - -1 and –1.2
   - -1.5 (repeated)
   - -1 + 10*j* and –1 –10*j*

   How are the real and imaginary parts of these poles related to the solution (output) of the system?

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