Name ………………………………………. Course ……………………………..

Print and use this sheet in conjunction with ***MathinSite***'s '1st & 2nd Order Systems: Unforced Response' applet and tutorial sheets.

First & Second Order Linear Systems: Unforced Response

Theory Sheet

Learning Outcomes

After using the ***MathinSite*** *First & Second Order System: Unforced Response* applet and its accompanying tutorial and theory sheets you should

- be aware of examples of engineering first and second order linear systems, their governing differential equations and their solution types
- be able to model some of these situations mathematically
- have developed, through experimentation with the applet, an understanding of how the different types of solution respond to changes in the *solution*'s parameters
- be able to answer "what if ...?" questions about the systems' unforced response.

Prerequisites

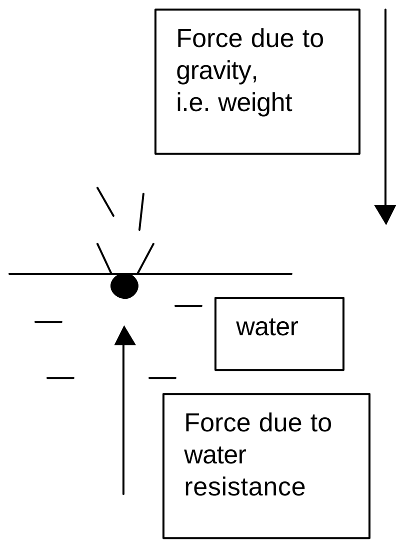
Before using the applet, this theory sheet and any accompanying tutorial sheets, familiarity with the following mathematics would be useful.

- The Straight Line
- The Exponential Function – including the notion of 'time constants'
- Trigonometrical Functions (in particular sines and cosines)
- Differentiation and Integration, and
- The solution of linear first and second-order differential equations

However, ***even without this knowledge***, just understanding how the systems respond can help in your appreciation of the mathematics involved. Applets covering most of the above mathematical topics can also be found on the ***MathinSite*** web site at http://mathinsite.bmth.ac.uk/html/applets.html.

Equations from Situations – Some Examples

What forces act upon a heavy ball as it falls through water? First, there is the gravitational force acting vertically downwards, and secondly, the force due to the resistance of the water vertically opposing the motion as the ball drops. The retarding force due to the water resistance can be found experimentally to be approximately proportional to the square of the velocity of the ball. So the net force downwards, using Newton's Second Law, is given by



$$m\frac{dv}{dt} = mg - mkv^2$$

where m is mass, v is velocity, t is time, g is the acceleration due to gravity and k is the constant of proportionality for the water resistance term.

This can be rewritten (after dividing throughout by m) as:

$$\frac{dv}{dt} + kv^2 = g$$

Unfortunately, owing to the presence of the v^2 term, this differential equation is *not* linear. However, as a rough-and-ready approximation (and only ever as such) the force due to water resistance can be considered as proportional to v only, in which case this equation becomes

$$\frac{dv}{dt} + kv = g \quad \text{(i)}$$

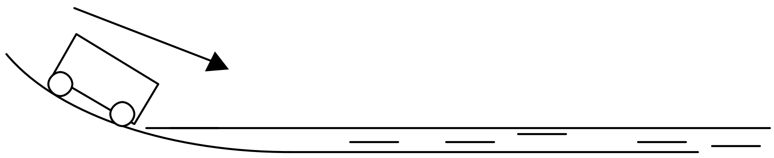
which is of the more general form:

$$a\frac{dy}{dt} + by = f(t)$$

that is, a ***linear*** first-order differential equation with constant coefficients.

Even so, our 'heavy ball' problem is still not quite in the format required here. The non-zero term on the right-hand side of equation (i) is a ***forcing*** term. The ball is 'forced' by gravity to continue to fall through the liquid despite a continuing retarding force due to liquid resistance (in such case a terminal velocity will be reached). We want here to remove the forcing term. What sort of situation would result in this?

Consider the following fairground ride as the truck passes through the water splash.



With, as a first approximation, v rather than v^2 again and now, with no gravity to accelerate or decelerate the truck horizontally, Newton's Second Law becomes:

$$\frac{dv}{dt} + kv = 0$$

This may have the overblown title of a linear, homogeneous first-order differential equation with constant coefficients, but the main point to consider here is that the truck, once it enters the water splash is ***unforced***; no *external* force drives it on – eventually it will slow to a stop (given enough length of track!). Since this motion ceases "after a short time", it is called the ***transient response***.

Other examples of first- and second-order systems can be found on ***MathinSite***. For example, the "LR Series Circuit" applet's first theory sheet (to be found on http://mathinsite.bmth.ac.uk/pdf/lrseries_theory1.pdf) shows that a series LR circuit with an applied emf, $e(t)$, has a governing first-order differential equation of the form:

$$L\frac{di}{dt} + Ri = e(t)$$

Here the *external* (to the basic LR system) applied emf is the forcing term; remove this and it becomes the ***unforced*** system

$$L\frac{di}{dt} + Ri = 0$$

You might wonder how you get any response out of an unforced system such as this; you don't - if the initial condition is zero! (The initial condition is the value of the system's dependent variable – here current, i – usually when $t = 0$.) In the unforced systems considered so far, to obtain a non-zero output it is necessary to have a non-zero initial condition. For example, in the water splash problem, there is no resultant velocity if the truck isn't moving initially. In the unforced LR circuit, no current will flow in the system if there was no current flowing in the first place. Note that as these are both *first*-order situations, they only require a *single* initial condition.

Examples of differential equations governing ***second-order systems*** can be found in the "Mass Spring Damper Systems" applet (whose theory sheet can be found on http://mathinsite.bmth.ac.uk/pdf/msdtheory.pdf) and the "LRC Series Circuit" (whose theory sheet can be found on http://mathinsite.bmth.ac.uk/pdf/lrcseries_theory1.pdf). These systems have the following associated differential equations:

$$M\frac{d^2y}{dt^2}+R\frac{dy}{dt}+ky=f(t) \qquad\qquad L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=e(t)$$

Each of these is derived in its associated theory sheet incorporating forcing terms. In the first, $f(t)$ is the externally applied force and in the second, $e(t)$ is the externally applied voltage. So the equivalent ***unforced*** systems will have governing equations:

$$M\frac{d^2y}{dt^2}+R\frac{dy}{dt}+ky=0 \qquad\qquad L\frac{d^2q}{dt^2}+R\frac{dq}{dt}+\frac{1}{C}q=0$$

As indicated previously, unforced systems will respond as long as they have non-zero initial conditions. Now, all second-order systems require ***two*** initial conditions (as opposed to simply the one required by first-order systems). For the mass-spring-damper, the two initial conditions are invariably the initial displacement, y, and the initial velocity, v (which is the rate of change of displacement, dy/dt). For the LRC series circuit governed by the above equation, they are the initial charge on the capacitor, q, and the initial rate of change of charge, dq/dt (i.e. the current, i).

Solving first and second-order differential equations of these types are covered in the theory sheets of the previously mentioned applets and so will not be covered here. However, a discussion of the type of solution obtained from a first-order system is to be found in an appendix to this document.

General Equations and Solutions

First-order Systems

The unforced first-order differential equations obtained from real-world situations considered here are all of the same general type:

$$a\frac{dy}{dt}+by=0$$

The solution of any general equation of this type is *always* going to be of the form:

$$y=Ae^{-\frac{b}{a}t}$$

Re-labelling b/a as k, results in the form of this output used in the accompanying applet, namely:

TABLE 1: Output/Response/Solution of the first-order system
$y=Ae^{-kt}$

For an indication how this solution was obtained, using two different methods, see the particular case of the solution of an LR circuit equation in the LR Series Circuit Theory Sheet.

In general then, for first-order systems such as those considered here, y, the time-dependent response varies exponentially. In any real-world first-order system without feedback, the constant coefficients a and b will necessarily both be positive (or perhaps for b, zero), in which case the system response will always be an exponentially decaying output (or constant, if $b = 0$). The constant, A, is effectively the 'constant of integration' and can be found using the initial condition. In fact, A ***is*** the initial condition (since $y = A$ when $t = 0$).

Second-order Systems

Unforced second-order differential equations obtained from real-world situations considered here are all of the same type:

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

The solution of any general equation of this type is *always* going to take one of the following three forms depending on the relative magnitude of b^2 against $4ac$:

TABLE 2: Output/Response/Solution of the second-order system		
If …	**Solution is …**	**Situation …**
$b^2 > 4ac$	$y = Ae^{-k_1t} + Be^{-k_2t}$	Heavy damping
$b^2 = 4ac$	$y = (A + Bt)e^{-kt}$	Critical damping
$b^2 < 4ac$	$y = e^{kt}\left(A\cos pt + B\sin pt\right)$	Light damping

Again, re-labelling occurrences/combinations of a, b and c in solutions (output) obtained from the second-order differential equation results in the form of this output used in the accompanying applet, as given in the above table.

Note that the output of both first- and second-order unforced systems such as these *always tend to a steady-state value of zero* - apart from the exceptional cases when some, or all, of the k values are zero.

<u>So what do typical responses look like?</u>

The two tables above give the type of response exhibited by both types of system – mathematically! Do you know what these responses look like graphically?

The associated applet allows the user to vary the parameters A, B, k, k_1, k_2 and p in such solutions to determine visually how each affects system response. Since all types of response encountered here are exponential and/or sinusoidal in nature, this applet can be considered as the 'applied mathematics' extension of the "Exponential Function" and "Trigonometrical Functions" applets/worksheets - also available from ***MathinSite*** on http://mathinsite.bmth.ac.uk/html/applets.html.

Exercises

1. Use either (both?) the separation of the variables method or Laplace Transforms to solve the first-order differential equation

$$a\frac{dy}{dt} + by = 0 \text{ given that } y = y_o \text{ when } t = 0.$$

How does your solution relate to the solution given in Table 1?

(Part answer: A is y_0)

2. Use any method you know to find the *general* solution of the second-order differential equation

$$a\frac{d^2y}{dt^2} + b\frac{dy}{dt} + cy = 0$$

given that

$$y = y_o \text{ and } dy/dt = y_1 \text{ when } t = 0$$

for the cases when

(i) $b^2 > 4ac$
(ii) $b^2 = 4ac$
(iii) $b^2 < 4ac$

giving your answers in terms of the variables y and t and the parameters a, b and c.

3. From the answers you have obtained in Q. 2, find the values of A, B, k, k_1, k_2 given in Table 2 in terms of any, or all, of a, b, c, y_o and y_1.

Appendix

Analysis of different parts of the solution of a forced LR Circuit equation

An LR Series Circuit is a 'linear system', that is, it behaves according to the rule "what goes in, comes out". So a sinusoidal voltage as input, for example, results in a sinusoidal current as output. However, before the circuit settles down to a ***steady state*** sinusoidal output, it usually exhibits an exponential response (the ***transient*** – or 'short lived' - part of the response).

The LR differential equation's solution comes in two parts:

$$i = Ae^{-\frac{Rt}{L}} + f(t)$$

The **first part**, $Ae^{-\frac{Rt}{L}}$ is called the **Complementary Function** (CF) and, containing R and L, results from the system (circuit) itself. Here, $-R/L$ is always negative (since $R > 0$ and $L > 0$), so the CF is *always* exponential decay for non-trivial cases. A can be negative or positive; if it is positive, exponential decay occurs, if it is negative then exponential growth to a limit occurs. (For further information see/use the 'Exponential Function' applet from ***MathinSite***.)

Since $Ae^{-\frac{Rt}{L}}$ eventually decays away, this part of the solution is called the ***transient***.

The **second part**, $f(t)$ is called the **Particular Integral** (PI) and results from, and takes the same form as, the forcing, applied voltage (the input to the system). So, if the applied voltage is a sinusoid, the PI will also be a sinusoid (possibly a mix of sines **and** cosines); if the input is ae^{pt} then the PI is Be^{pt} (or $(Bt + C)e^{pt}$ under certain conditions).

$f(t)$ is the ***steady state*** part of the response *and will always be zero in unforced systems*

The overall solution (current), $i(t)$, is the sum of the CF and PI.

Note that various mixes of values for the initial conditions, system parameters and applied voltage parameters ***can*** *result in any part of the full solution being zero.*

… and remember,
the PI (steady state output) is the same form as the input (forcing term)
… and so,
if the system is unforced, the PI part of the solution (response) is zero
… and so,
solutions of unforced systems only consist of the CF (the transient response)