

## The Trigonometrical Functions

### The Software Tutorial



#### Learning Outcomes

After using the accompanying "Trigonometric Functions" applet, the user should have:

- developed a deeper understanding of the trigonometric functions  $A \sin(\omega t + \phi)$ ,  $A \cos(\omega t + \phi)$  and  $A \tan(\omega t + \phi)$ ,
- gained a feel for the effect of changing the parameters,  $A$ ,  $\omega$  and  $\phi$ .
- gained sufficient expertise to answer "what if ...?" questions about the effect of changing the parameters of trigonometrical functions.

#### Introduction

Sine, cosine and tangent are three basic trigonometrical functions first encountered at secondary school. Although commonly used at this level, many students will not necessarily have plotted graphs of these functions - especially in the form  $y = A \sin(\omega t + \phi)$ , a form commonly used in engineering applications where time ( $t$ ) is usually the independent variable. The accompanying software allows users to 'play' with functions of this type and investigate what happens when the functions' parameters are changed.

$t$  and  $y$  are **variables**; on the graphs displayed using the software, as the value of  $t$  varies, so the  $y$  value varies also.

$t$  is called the *independent variable* (the horizontal axis on the graph) and  $y$  is called the *dependent variable* (the vertical axis). The value of  $y$  *depends on* the value of  $t$ .

The variable  $t$  is used here, rather than the more usual  $x$ , since, as indicated above, when these functions appear in engineering and science, the independent variable is usually time.

$A$ ,  $\omega$  and  $\phi$  are the function's **parameters**. Choosing a value for each of  $A$ ,  $\omega$  and  $\phi$  will result in one particular trigonometric curve. A new set of values for  $A$ ,  $\omega$  and/or  $\phi$  will result in a *different* trigonometric curve (but of the same type).

The graphs of  $y = A \sin(\omega t + \phi)$  and  $y = A \cos(\omega t + \phi)$  have the well-known shape of the sine wave - an oscillating curve. Since tangent is not defined for  $90^\circ$  ( $\frac{\pi}{2}$  radians), and various other multiples of  $90^\circ$ , the tangent curve will not take the same form as the others.

If  $y$  is a function of (i.e.  $y$  depends on)  $t$ , written  $y = f(t)$  in general, then here we can write  $f(t) = A \sin(\omega t + \phi)$ . In this case,  $f(t)$  is called a **trigonometric function**.

Trigonometric functions have important applications in the mathematical analysis of topics in science, technology and business studies – in particular in cases where **oscillations**, or events that occur in nature with strict **periodicity** occur. For example,

- a non-damped mass on the end of a spring performs Simple Harmonic Motion; a graph of the mass' displacement against time will show the characteristic **sinusoidal** wave shape of the sine/cosine waves,

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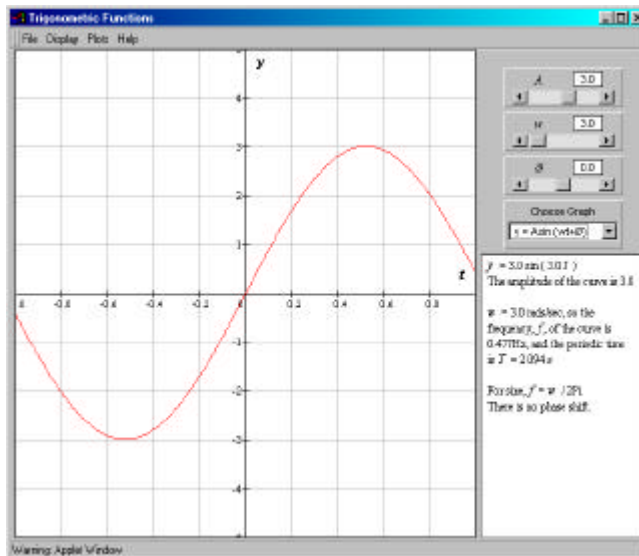
- a graph of the lighting-up times (the time in the evenings when city streetlights are switched on) against the 52 weeks of the year will exhibit the sinusoidal shape.

### Loading the Software

The "Trigonometric Functions" applet should be run from the same web page from which this tutorial worksheet was downloaded.

### Running the applet

When loaded, you will see an applet window similar to the one shown here. By default, the software loads the sine function with parameter values  $A = 3$ ,  $\omega = 4$  and  $f = 0$ . With these default values, the relevant graph (in this case,  $y = 3 \sin 4t$ ) is displayed in the main graphics area as indicated in the above screenshot.



The three slider bars on the right of the applet window allow the user to change each of the parameters (here  $A$ ,  $\omega$  and  $f$ ). Clicking the arrow boxes at either end of the slider bar effect *small changes* in the parameter values. Clicking in the main part of the box of the slider bar makes *larger changes*. *Clicking, holding and moving the slider in the slider bar can make any sort of change.*

***Changing parameter values by moving the sliders is the only means of user-input.***

The applet gives you the option of showing single plots or multiple plots (under 'Plots' in the pull down menu structure at the top of the applet window). It also allows you to 'drag' the plot around (to see areas not covered by the original window on the plot) by holding down the mouse button whilst at the same time moving the mouse over the plot. The plot can be centred at any time using the 'Centre on Origin' option, under 'Display'. You can reset the software at any time to the default values (those used when the program loads, including the default function, sine) using 'Reset' from the 'Plots' menu.

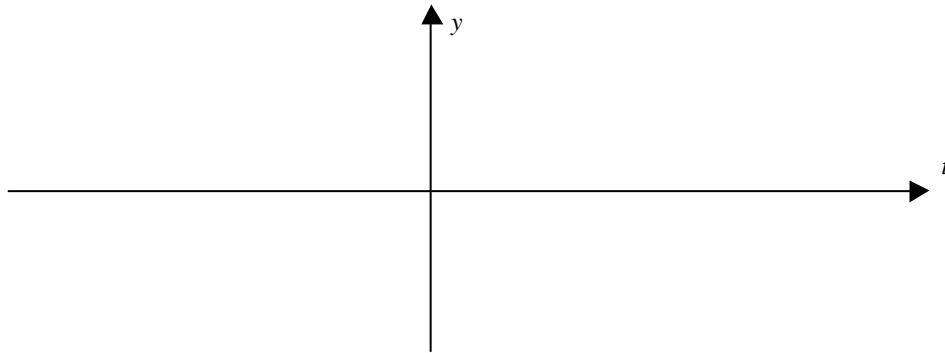
## The Tutorial

### The effect of changing $A$

- With the default values  $A = 3$ ,  $\omega = 4$  and  $f = 0$ , sketch the graph that you see on-screen on the axes on the next page and write its equation alongside.
- Choose "Multiple Plots" from the "Plots" pull-down menu and change the value of  $A$  to  $A = 4$  by clicking once *inside* the  $A$  slider bar to the right of the slider itself.
- Repeat so that your plot shows also the graphs for  $A = 5$ . Now go back through the case  $A = 4$  and  $A = 3$  by clicking once *inside* the  $A$  slider bar to the left of the slider itself and bring up the plots for the cases  $A = 2$  and  $A = 1$  also.

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- Superimpose all the curves on the axes below, labelling each line with its equation. Note carefully where each plot passes through the  $t$  axis.



Where did each curve cross the  $y$ -axis?

.....

What was special about the positions of where the curves crossed the  $t$  axis?

.....

What was the effect on the curves of increasing  $A$  from 1 to 5?

.....

On your plots, what distance is measured by the value of  $A$ ?

.....

What do you think the plot would look like if  $A = 0$ ? What is its equation in this case?

..... Try it now!

Why do you think the software doesn't allow you to use negative values of  $A$ ?

.....

A clearer picture of the effect of changing  $A$  can be seen if you:

- Reset the graphic display to the default setting by choosing "Reset" from the "Plots" pull-down menu.
- Choose "Single Plot" from the "Plots" pull-down menu (or "Clear Multiple Plots")
- With the default values  $A = 3$ ,  $w = 4$  and  $f = 0$ , move the  $A$  slider to zero.
- Choose "Multiple Plots" from the "Plots" pull-down menu (if not already active).
- Press and hold the arrow at the right-hand end of the  $A$  slider and watch the graphics. Cool, huh?

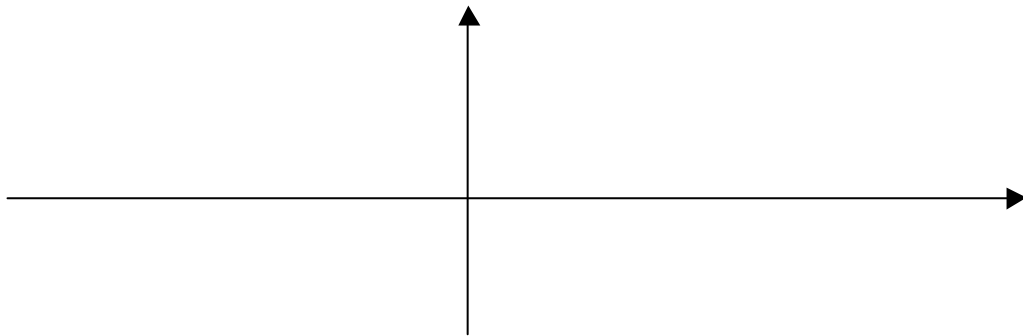
*'A' is called the AMPLITUDE of the Sine wave*

Now have a look at the cosine curve:

- Reset the graphic display to the default setting by choosing "Reset" from the "Plots" pull-down menu.

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- Choose "Single Plot" from the "Plots" pull-down menu (or "Clear Multiple Plots").
- If not already chosen, choose "Multiple Plots" from the "Plots" pull-down menu and change the function from  $y = A \sin(\omega t + f)$  to  $y = A \cos(\omega t + f)$  by clicking once on the 'down arrow' on the box on the right currently showing  $y = A \sin(\omega t + f)$  and select  $y = A \cos(\omega t + f)$ . The new plot has the same parameters (you didn't change these) and so shows  $y = 3 \cos 4t$ .
- Superimpose each of these two curves on the axes below, labelling each with its equation. (You may want to see how both curves appear outside of the range shown in the graphics window. This can be achieved by clicking and holding down the mouse button in the graphics area and moving it left or right. Note that the y-axis can always be restored by choosing "Centre on Origin" from the "Display" menu.)



List three aspects of the two curves that are the same (shape, distances involved, etc):

.....

List any differences between the two curves.

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Why do you suppose then that this tutorial does not ask you to repeat the exercises done above to see the effect of changing  $A$  for the cosine wave?

.....

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Now have a look at the tangent curve:

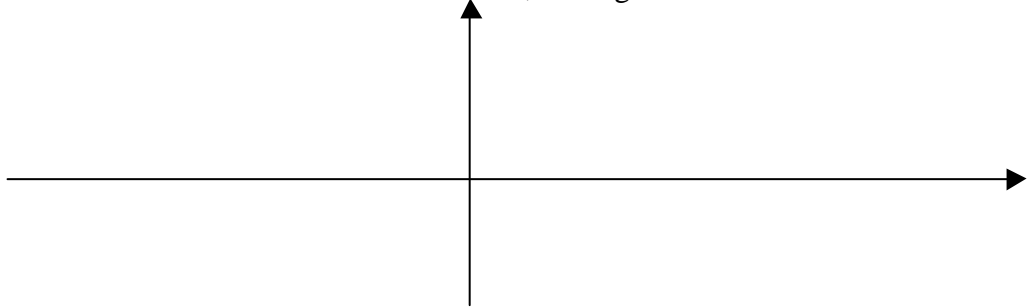
- Again reset the graphic display to the default setting by choosing "Reset" from the "Plots" pull-down menu.
- Choose "Single Plot" from the "Plots" menu (or use "Clear Multiple Plots")
- With the default values  $A = 3$ ,  $\omega = 4$  and  $f = 0$ , change the function from  $y = A \sin(\omega t + f)$  to  $y = A \tan(\omega t + f)$  by clicking once on the 'down arrow' on the box on the right and this time select  $y = A \tan(\omega t + f)$ . The new plot has the same parameters (you didn't change these) and so shows  $y = 3 \tan 4t$ .
- Sketch and label this graph on the following axes on the next page.
- Again you may want to see how both curves appear outside of the range shown in the graphics window. This can be achieved by clicking and holding down the mouse button

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in the graphics area and moving it left to right or vice versa or, this time, up and down.

Try it. Now bring back into view the y-axis so the origin can be seen (choose "Centre on Origin" from the "Display" menu).

- Click once to the left of the  $A$  slider so that  $A$  is 2, do it again so  $A$  is 1.



What effect did changing  $A$  have this time?

.....

To see this effect more clearly, place the mouse cursor on the arrow at either end of the  $A$  slider bar and keep it held down. Watch the graphics as  $A$  is changed. Trying it with "Multiple Plots" perhaps gives a better idea. Now do the same but increasing  $A$ .

The blue vertical lines superimposed on the tangent graph are called *asymptotes*. Move the curve vertically and state what happens to the curve for 'extreme'  $y$  values?

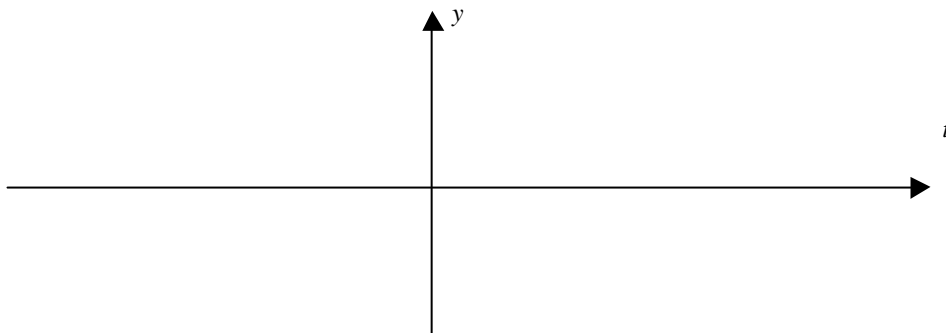
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So, in what way does changing  $A$  affect the tangent curve? Try here to express this in such a way that it could also fit the sine and cosine curves.

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### **The effect of changing $W$**

- Reset the graphic display to the default setting by choosing "Reset" from the "Plots" pull-down menu.
- Choose "Single Plot" from the "Plots" menu (or "Clear Multiple Plots").
- With the default values  $A = 3$ ,  $W = 4$  and  $f = 0$ , sketch the graph that you see on-screen on the axes below and write alongside this line its equation.
- Ensure "Multiple Plots" is active and change the value of  $W$  to  $W = 5$  then  $W = 6$  by clicking twice *on the right-hand end arrow* on the  $W$  slider.
- Superimpose these three curves on the axes below, labelling each line with its equation. Note carefully where each plot passes through the  $t$  axis.



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What was the effect on the curves of increasing  $W$  from 4 to 6?

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What happened to the amplitude?

.....

A clearer picture of the effect of changing  $W$  can be seen if you:

- Reset the graphic display to the default setting by choosing "Reset" from the "Plots" pull-down menu.
- Choose "Single Plot" from the "Plots" pull-down menu (or "Clear Multiple Plots")
- With the default values  $A = 3$ ,  $W = 4$  and  $f = 0$ , move the  $W$  slider to zero at the left-hand end of the slider.
- Press and hold the arrow at the right-hand end of the  $W$  slider and watch the graphics. Cool again, huh?

(You might prefer to use "Multiple Plots" to see this better.)

Why do you think the software doesn't use negative values of  $W$ ?

.....

*'W' is called the ANGULAR VELOCITY of the Sine wave  
and is measured in radians per second.  
W gives a measure of how 'frequently' the curve repeats itself  
so is related to FREQUENCY  
- as W increases, the frequency increases*

You could now have a look at the cosine curve to see if the effect is the same. Rather than plot several curves here, play with the slider for  $W$  as outline just previously, i.e. move the  $W$  slider to zero, clear any multiple plots and hold the right-hand end slider down. Try this with the tangent function too.

- In what way does changing  $W$  affect all three curves? Try to express this in such a way that it could also fit the sine, cosine and tangent curves.

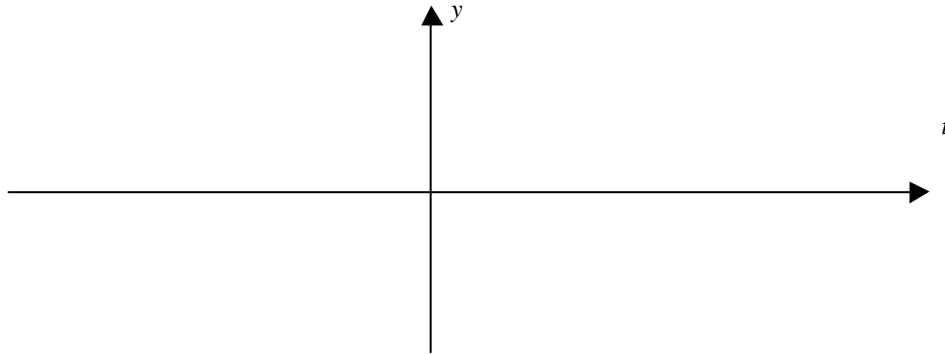
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### The effect of changing $f$

- Reset the graphic display to the default setting by choosing "Reset" from the "Plots" pull-down menu.
- Choose "Single Plot" from the "Plots" menu (or "Clear Multiple Plots").
- With the default values  $A = 3$ ,  $W = 4$  and  $f = 0$ , sketch the graph that you see on-screen on the axes below and write alongside this line its equation.

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- Ensure "Multiple Plots" is active. Change the value of  $f$  to  $f = 0.25$ , giving the equation  $y = 3\sin(4t + 0.25)$ , and then to  $f = 0.5$  by clicking successively on the arrow at the *right-hand* of the  $f$  slider bar.
- Superimpose these three curves on the axes below, labelling each line with its equation. Note carefully where each plot passes through the  $t$  axis.



What was the effect on the curves of increasing  $f$  from 0 through 0.25 to 0.5?

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What happened to the amplitude and the frequency?

.....

Increasing  $f$  moved the curve in which direction?

.....

Now try the same process but change  $f$  by any amount in the *negative direction*.

Increasing  $f$  in the negative direction moved the curve in which direction?

.....

A clearer picture of the effect of changing  $f$  can be seen if you:

- Reset the graphic display to the default setting by choosing "Reset" from the "Plots" pull-down menu.
- Choose "Single Plot" from the "Plots" pull-down menu (or "Clear Multiple Plots")
- With the default values  $A = 3$ ,  $W = 4$  and  $f = 0$ , move the  $f$  slider to the left and right by clicking and holding on the slider itself.

(You might prefer to use "Multiple Plots".)

*' f ' is called the PHASE ANGLE  
and is invariably, in an engineering context, measured in RADIANS*

You could now have a look at the cosine curve to see if the effect is the same. Rather than plot several curves here, play with the slider for  $W$  as outlined just previously. Try this with the tangent function too.

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- In what way does changing  $f$  affect all three curves? Try to express this in such a way that it could also fit the sine, cosine and tangent curves.

.....

**So changing the value of  $A$  alone expands or compresses the curves vertically,  
changing the value of  $W$  expands or compresses the curves horizontally  
and changing the value of  $f$  alone moves the curves left or right.**

### Some Useful Theory

You probably noticed the repetitive nature of all three curves; they are all *periodic*.

Both sine and cosine repeat themselves every time that  $Wt$  goes 'full circle', i.e.  $360^\circ$ , or  $2p$  radians in radian measure. The tangent curve repeats itself however after only  $180^\circ$ , or  $p$  radians. You can see this clearly if you reset the software, choose multiple plots and then  $y = A \tan(Wt + f)$  so you can superimpose the curves  $y = 3 \sin 4t$  and  $y = 3 \tan 4t$  on the same axes. Note how tangent repeats twice for sine's once.

The *time* that this takes, usually denoted  $T$  and called the *periodic time*, is therefore, for sine and cosine, the time when  $WT = 2p$

*So for the sine and cosine curves,  $T = \frac{2p}{W}$ , for the tangent curve,  $T = \frac{p}{W}$*

How frequently the curves repeat is measured by the frequency. If the sine wave is repeating at 1 cycle per sec (or 1 hertz), it is executing  $2p$  radians every second. So a frequency of  $f$  cycles per second will be equivalent to  $2pf$  radians per second - and this is  $W$ .

*Angular velocity  $W$  is directly proportional to the frequency of the oscillations,  $f$ ,  
and is given by  $W = 2pf$  for sine and cosine*

If the phase angle is  $f$  radians, then  $y = A \sin(Wt + f)$  is said to *lead* the graph of  $y = A \sin Wt$  by  $f$  radians (remember, the curve is shifted left as  $f$  increases).

If the phase angle is  $f$  radians, then  $y = A \sin(Wt - f)$  is said to *lag* the graph of  $y = A \sin Wt$  by  $f$  radians (remember, the curve is shifted right as  $f$  decreases).

Neither of these facts is much use on a graph where the independent variable is measured in terms of time (seconds, perhaps). However, the shift on the time axis can be found very simply by putting  $Wt + f = 0$ , i.e.

**The time phase shift on any of these trigonometrical curves is given by  $t_{ps} = -\frac{f}{W}$**



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Note that this tells you when  $f$  is positive, the time shift will be negative and when  $f$  is negative, the time shift will be positive.

Try this out using the applet. Use "Reset" to give the default sine wave with  $A = 3$ ,  $W = 4$  and  $f = 0$ , and then change  $f$  from 0 to -1.5. You should see a time shift to the right of

$$t_{ps} = -\frac{f}{w} = -\frac{-1.5}{4} = 0.375 \text{ sec.}$$

This is called a "phase lag" since this curve "starts" 0.375 seconds *later* than the basic curve for which  $f$  is zero.

Note that phase lead (or phase lag) measured as an angle in radians is dependent solely on  $f$ , but as a time shift is dependent on both  $W$  and  $f$  (but independent of the value of  $A$ ).

### Exercises

1. *Without* using the software, sketch<sup>Ⓢ</sup> the following trigonometrical curves. It is to your advantage if you note the values of  $A$ ,  $W$  and  $f$  and hence determine the periodic time and the phase shift in terms of time using the theory on the previous page.

$$y = 4 \sin t \qquad y = 4 \sin(t + 0.1)$$

$$y = 3 \cos 4t \qquad y = 3 \cos(4t - 0.8)$$

$$y = 4 \tan 2t \qquad y = 4 \tan\left(2t + \frac{\pi}{2}\right)$$

Verify your sketches using the software.

2. Set up the software so that you are using "Single Plot" with the default values and function ( $y = 3 \sin 4t$ ). Set  $W$  to zero. You should see a straight line along the  $t$ -axis. Why? What is the equation for this case? Grab and move the  $W$  slider to the left-hand end of the slider bar. Watch as you now press and hold the arrow at the right-hand end of the  $W$  slider. What's happening? Why? What equation are you successively plotting in each case?
3. What are the similarities and the differences between the curves  $y = 3 \sin(2t - 2)$ ,  $y = 3 \sin(3t - 3)$  and  $y = 3 \sin(4t - 4)$ ? Use the software to help you check your answer, noting particularly the phase shift on the time axis in each case. Why should this be so since the phase angle is different for each case?

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<sup>Ⓢ</sup> Here 'sketch' means roughly draw the axes with the curves superimposed. You should label the axes and each graph and indicate any important points (such as the  $y$ -intercept, the time on the  $t$  axis for one complete period of the sin, cos or tan curve and the phase shift in time). 'Plot', on the other hand, means 'plot points accurately on a properly scaled and labelled graph' using data from a table of values. You should label the graphs and important points in either case.